PMATH 763, Winter 2017

Assignment #3 Due: March 7.

1. **(a)** Let $\mathfrak{m}$ be a non-abelian finite dimensional $\mathbb{R}$-Lie algebra for which $[\mathfrak{m}, \mathfrak{m}]$ is abelian. Such a Lie algebra may be called *metabelian*. Show that $\mathfrak{m}$ contains at least one of the following 3 types real Lie subalgebras:

$$\begin{aligned} \mathfrak{f} &= \mathrm{span}_{\mathbb{R}}\{X, Y\},\ [X, Y] = Y \text{ (affine motion)} \\ \mathfrak{g}_\theta &= \mathrm{span}_{\mathbb{R}}\{X, Y_1, Y_2\},\ [X, Y_1] = \theta Y_1 - Y_2,\ [X, Y_2] = Y_1 + \theta Y_2, (\theta \in \mathbb{R}),\ [Y_1, Y_2] = 0 \\ \mathfrak{h} &= \mathrm{span}_{\mathbb{R}}\{X, Y, Z\},\ [X, Y] = Z,\ [X, Z] = 0 = [Y, Z] \text{ (Heisenberg).} \end{aligned}$$

[Hint. Let $X \in \mathfrak{g} \setminus [\mathfrak{m}, \mathfrak{m}]$ and consider the eigenvalues of $\mathrm{ad}X \in \mathcal{L}(\mathfrak{m})$; consider cases of a non-zero real eigenvalue, a complex eigenvalue, and only zero as an eignenvalue. Of course if $\mathrm{ad}X$ admits only 0 as an eigenvalue, it is a nilpotent operator.]

[It is nice, though not necessary to (a), to note concrete realisation of each of these 3 classes:

$$\mathfrak{f} = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x, y \in \mathbb{R} \right\},\ \mathfrak{g}_\theta = \left\{ \begin{bmatrix} \theta x & x & y_1 \\ -x & \theta x & y_2 \\ 0 & 0 & 0 \end{bmatrix},\ x, y_1, y_2 \in \mathbb{R} \right\},\ \mathfrak{h} = \mathfrak{t}_3^0(\mathbb{R}).]$$

(b) Deduce that any finite dimensional Lie algebra $\mathfrak{g}$, which contains a non-abelian solvable Lie algebra $\mathfrak{r}$, contains of one the 3 types of Lie algebras from (a).

2. If G is a matrix Lie group, let G' denote the closed subgroup generated by group commutators: $ghg^{-1}h^{-1}$, $g, h \in G$. Then let $G^{(0)} = G$ and for k in $\mathbb{N}$ let $G^{(k)} = (G^{k-1})'$. We say that G is *solvable* of $G^{(m)} = \{I\}$ for some m.

(a) Show that if G is connected with $\mathfrak{g} = \mathrm{Lie}(G)$, then $\mathrm{Lie}(G') \supseteq [\mathfrak{g}, \mathfrak{g}]$. Deduce that if G is solvable, then so too must be the Lie algebra $\mathfrak{g}$.

[Hint. Like in the proof that $\mathrm{Lie}(G)$ is closed under addtition, first show that there is a continuous function R of t, X, Y (small $|t|$) for which $\exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = \exp(t^2[X, Y] + t^4R(t, X, Y))$.]

(b) Deduce that if $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is any representation of a solvable connected Lie group, then there is a basis B of $\mathbb{C}^n$ for which $[\rho(G)]_B \in \mathrm{T}_n(\mathbb{C})$.

3. **(a)** Show that the only proper Lie ideals of $\mathfrak{gl}_n(\mathbb{F})$ are $\mathbb{F}I$ and $\mathfrak{sl}_n(\mathbb{F})$.

(b) Deduce that $\mathfrak{sl}_n(\mathbb{F})$ is a *simple* Lie algebra, i.e. it admits no proper Lie ideals.

4. Show that the Killing form on $\mathfrak{so}(n)$ is given by

$$B(X,Y) = (n-2)\mathrm{Tr}(XY).$$

[With respect to $((X,Y)) = \mathrm{Tr}(XY^T)$ we have that $(X_{ij} = \frac{1}{\sqrt{2}}(E_{ij} - E_{ji}))_{1 \leq i < j \leq n}$ is an orthonormal basis. Compute $B(X,Y) = \sum_{1 \leq i < j \leq n} ((\mathrm{ad}X \circ \mathrm{ad}Y(X_{ij}), X_{ij}))$.]

5. A finite dimensional $\mathbb{R}$-Lie algebra is *reductive* if it is of the form $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$, i.e. Lie algebra direct sum: $\mathfrak{g} = \mathfrak{a} + \mathfrak{s}$ with $\mathfrak{a} \cap \mathfrak{s} = \{0\} = [\mathfrak{a}, \mathfrak{s}]$, where $\mathfrak{a}$ is an abelian Lie alebra and $\mathfrak{s}$ is a semisimple Lie algebra.

   Let $\mathfrak{z} = \mathrm{Z}(\mathfrak{g})$ denote the centre and $\mathfrak{r} = \mathrm{rad}(\mathfrak{g})$ denote the radical$^{(*)}$, of a Lie algebra $\mathfrak{g}$.

   Show that the following are equivalent:

   (i) $\mathfrak{g}$ is reductive;

   (ii) every abelian ideal of $\mathfrak{g}$ lies in $\mathfrak{z}$, and $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{z} = \{0\}$;

   (iii) $\mathfrak{r} = \mathfrak{z}$; and

   (iv) $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is *completely reducible*, i.e. there is a family $\mathfrak{g}_1, \dots, \mathfrak{g}_m$ of subspaces of $\mathfrak{g}$ for which $\mathfrak{g} = \bigoplus_{j=1}^m \mathfrak{g}_j$, and each $\mathfrak{g}_j$ is invariant and irreducible for ad.

   [Consider (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii); (i)&(ii) $\Leftrightarrow$ (iv). Each $\mathfrak{g}_j$ in (iv) is either a one-dimensional ideal or a semi-simple ideal of $\mathfrak{g}$.]

   $(*)$ I missed this in class. The *radical* is the largest solvable ideal in $\mathfrak{g}$. Why does this exist? We saw that the sum $\mathfrak{i} + \mathfrak{j}$ of two solvable ideals in $\mathfrak{g}$ is itself a sovable ideal. Hence let $\mathfrak{r}$ be a solvable ideal of maximal dimension. Then if $\mathfrak{i}$ is another sovable ideal we see that $\mathfrak{i} + \mathfrak{r}$ is a solvable ideal, which must be of the same dimension as $\mathfrak{r}$, whence equal to $\mathfrak{r}$, so $\mathfrak{i} \subseteq \mathfrak{r}$.