PMATH 763, Winter 2017

Assignment #1 Due: January 24.

1. **(a)** Show that $\mathrm{GL}_n(\mathbb{F})$ is dense in $\mathrm{M}_n(\mathbb{F})$. Hence deduce that $\mathrm{GL}_n(\mathbb{F})$ is not complete in the metric given by the norm $\|\cdot\|$.

   (b) (Polar decomposition.) If $A \in \mathrm{M}_n(\mathbb{R})$, show that there exists $u \in \mathrm{O}(n)$ and P in $\mathrm{M}_n(\mathbb{R})$ for which $P^T = P$ and $(Px, x) \geq 0$ for all x in $\mathbb{R}^n$, such that $A = uP$. Show that if A is not invertible, then the decomposition is not unique.

   (c) Show that if $(g_k)_{k=1}^{\infty}$ is a sequence in $\mathrm{GL}_n(\mathbb{F})$ which is *left Cauchy* in the sense that for every $\epsilon > 0$, there is k_ϵ such that for $k, \ell \geq k_\epsilon$ such that

   $$\|g_k^{-1} g_\ell - I\| < \epsilon$$

   then $\lim_{k\to\infty} g_k$ exists in $\mathrm{GL}_n(\mathbb{F})$.

   This gives us a sense in which $\mathrm{GL}_n(\mathbb{F})$ is "complete".

2. **(a)** Show that each of $\mathrm{SU}(n)$, $\mathrm{U}(n)$, $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$ are connected.

   (b) Recall that $\mathrm{T}_n(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}) : g_{ij} = 0 \text{ if } j < i\}$. Determine the number of connected components of this group.

   (c) Do the same for $\mathrm{T}_n(\mathbb{C})$.

(There are questions on the next page too.)

3. ("Principal branch" of matricial logarithm) Let

$$\mathcal{U} = \{X \in \mathrm{M}_n(\mathbb{F}) : X \text{ has no eigenvalues in } (-\infty, -1]\}.$$

For $X \in \mathcal{U}$ let

$$L(X) = X \int_0^1 (I + tX)^{-1} dt.$$

(a) Show that L is continuous on $\mathcal{U}$.

[First establish the easy inequality $\| \int_0^1 F(t)\, dt\| \leq \int_0^1 \|F(t)\|\, dt$ for continuous $F : [0,1] \to \mathrm{M}_n(\mathbb{F})$.]

(b) Show that if $\|X\| < 1$, then $X \in \mathcal{U}$ and $\exp(L(X)) = I + X$.

[Hint: verify uniform convergence of the series given by $(I + tX)^{-1}$, for $t \in [0,1]$, in order to justify interchange of limit and integral.]

Let $\mathrm{Sym}_n(\mathbb{R}) = \{H \in \mathrm{M}_n(\mathbb{R}) : H^T = H\}$.

(c) Show that if $p \in \mathcal{P}_n(\mathbb{R}) \cap \mathrm{Sym}_n(\mathbb{R})$ then $p - I \in \mathcal{U}$ and $L(p - I) \in \mathrm{Sym}_n(\mathbb{R})$ with $\exp(L(p-I)) = p$. Moreover, if $H \in \mathrm{Sym}_n(\mathbb{R})$ then $\exp(H) \in \mathcal{P}_n(\mathbb{F})$ with $L(\exp H - I) = H$.

Remark: The continuity of exp, then parts (a) and (c) show that $\exp : \mathrm{Sym}_n(\mathbb{R}) \to \mathcal{P}_n(\mathbb{R}) \cap \mathrm{Sym}_n(\mathbb{R})$ is a homeomorphism. A similar proof shows that the same is true for $\exp : \mathrm{Herm}(n) \to \mathcal{P}_n(\mathbb{C})$ where $\mathrm{Herm}(n) = \{H \in \mathrm{M}_n(\mathbb{C}) : H^* = H\}$.

(d) Show that $\exp(L(g - I)) = g$ for any g in $\mathrm{GL}_n(\mathbb{R})$ such that $g - I \in \mathcal{U}$.

[Hint: Show that the real variable function $F(s) = \exp(L(sX))$ is analytic on some open interval containing $[0,1]$. Then show that the set $\{s \in [0,1] : F^{(n)}(s) = 0 \text{ for all } n\}$ is both (relatively) open and closed.]

4. Let $f(t) = \begin{cases} e^{1/(t^2-1)} & \text{if } |t| < 1 \\ 0 & \text{if } |t| \geq 1 \end{cases}$. Show that $f \in \mathcal{C}^\infty(\mathbb{R})$.

Remark: At either of $t = \pm 1$, f admits trivial Taylor series and thus is not analytic on all of $\mathbb{R}$.