PMATH 763, Winter 2017

Assignment #2 Due: February 9.

1. Two elements g, g' of a group G are called *conjugate* if there is $h \in G$ such that $hgh^{-1} = g'$.

   (a) Show that $\exp(\mathfrak{u}(n)) = \mathrm{U}(n)$ and $\exp(\mathfrak{su}(n)) = \mathrm{SU}(n)$.

   [Hint: We want each element of the group expressable as a single exponential. Use diagonalisation.]

   (b) Show that every element $g \in \mathrm{SL}_2(\mathbb{R})$ is conjugate to one of

$$\begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix}, \quad \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & t \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

   where $a \in \mathbb{R} \setminus \{0\}$, $t \in \mathbb{R}$ and $\theta \in [0, 2\pi)$. Then show that

$$\exp(\mathfrak{sl}_2(\mathbb{R})) = \{g \in \mathrm{SL}_2(\mathbb{R}) : \mathrm{Tr}(g) > -2\} \cup \{-I\}.$$

   [To see the conjugacy, consider the eigenvalues of G; conduct a change of basis.]

   (c) Show that every element $g \in \mathrm{SL}_2(\mathbb{C})$ is conjugate to one of

$$\begin{bmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

   where $\alpha \in \mathbb{C} \setminus \{0\}$. Determine if $\exp : \mathfrak{sl}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})$ is surjective.

2. (Heisenberg group and Lie algebra) Let $H = \mathrm{T}_3^0(\mathbb{R})$ and $\mathfrak{h} = \mathfrak{t}_3^0(\mathbb{R})$.

   (a) Show that $\exp : \mathfrak{h} \to H$ is a bijection.

   (c) On $H_r = \mathbb{R}^2 \times \mathrm{U}(1)$ define the product

$$(x, y, \zeta) \cdot (x', y', \zeta') = (x + x', y + y', e^{ixy'}\zeta\zeta').$$

   Show that $\varphi : H \to H_r$ given by

$$\varphi\left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}\right) = (x, y, e^{iz})$$

   is a continuous group homomorphism. [H_r is called the "reduced Heisenberg group" in signal processing literature.]

(c) Let $X = E_{12}, Y = E_{23}$ and $Z = E_{13}$ (matrix units) in $\mathfrak{h}$, and $\alpha : \mathfrak{h} \to \mathfrak{gl}_n(\mathbb{C})$ be a Lie algebra representation. Show that $\alpha(Z)$ is necessarily nilpotent in $\mathrm{M}_n(\mathbb{C})$.

[Hint: Show that any eigenspace $\mathcal{V}$ of $\alpha(Z)$ must be α-invariant, and hence $\alpha(Z)|_{\mathcal{V}} = \alpha([X, Y])|_{\mathcal{V}}$ must be trace zero in $\mathcal{L}(\mathcal{V})$.]

(d) Show that the is no injective representation $\pi : H_r \to \mathrm{GL}_n(\mathbb{C})$ for any n. Hence H_r is not a matrix Lie group. [The reduced Heisenberg group seems to have every right to call itself a "Lie group", but it is not a matrix Lie group.]

[Hint: Consider $\rho = \pi \circ \varphi$, which has discrete kernel. Hence use (c) to show that $\rho(\exp tZ) = \exp(td\rho(Z))$ can be I for some t in $\mathbb{R} \setminus \{0\}$ only if $d\rho(Z) = 0$.]

3. **(a)** Let $G \subset \mathrm{GL}_n(\mathbb{R})$ be a connected abelian matrix Lie group. Show that $\mathfrak{g} = \mathrm{Lie}(G)$ is abelian in the sense that $[X, Y] = 0$ for X, Y in $\mathfrak{g}$. Furthermore, $\exp : \mathfrak{g} \to G$ is a covering map. [Here the additive group $\mathfrak{g}$ may be regarded as a matrix Lie group, say, as $\left\{ \begin{bmatrix} I_n & X \\ 0 & I_n \end{bmatrix} : X \in \mathfrak{g} \right\} \subseteq \mathrm{GL}_{2n}(\mathbb{R})$.]

(b) Conclude that a connected abelian Lie group is isomorphic to $\mathbb{R}^k \times \mathrm{U}(1)^l$ for some $k, l \in \{0\} \cup \mathbb{N}$.

[Hint: It is sufficient to see that any discrete subgroup Γ of $\mathbb{R}^d$ is of the form $\Gamma = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_m$ where $\{v_1, \ldots, v_m\}$ is a linearly independent subset of $\mathbb{R}^d$. Show that if $\{w_1, \ldots, w_m\}$ is a linearly independant subset of maximal size in Γ, then the subgroup $\Lambda = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_m$ is of finite index in Γ. You may use, without proof, the fact from algebra that any subgroup of $\mathbb{Z}^m$ is isomorphic to $\mathbb{Z}^k$ for some $k \leq m$.]

(c) Fix an irrational number ξ. Show that the 1-dimensional Lie algebra

$$\mathfrak{h} = \left\{ \begin{bmatrix} i2\pi t & 0 \\ 0 & i2\pi\xi t \end{bmatrix} : t \in \mathbb{R} \right\} \subset \mathfrak{gl}_2(\mathbb{C})$$

satisfies that

$$\exp : \mathfrak{h} \to K = \left\{ \begin{bmatrix} w & 0 \\ 0 & z \end{bmatrix} : w, z \in \mathbb{C}, |w| = 1 = |z| \right\}$$

is injective, has dense range but is not surjective. In particular, conclude that $\mathfrak{h}$ cannot be $\mathrm{Lie}(H)$ for any closed subgroup H of $\mathrm{GL}_2(\mathbb{C})$.

[Hint: Begin by showing that $\{k + \xi l : k, l \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Then show for every (r, s) in $\mathbb{R}^2$ that one can choose k, l and t so $|\xi t - s - l|$ is small with $t - r - k = 0$.]