Candidate Final Exam Questions

The questions on the final exam will be extracted nearly literally form this list. You are expected to know all definitions and notation.

1. **(a)** Prove *polar decomposition* in $\mathrm{GL}_n(\mathbb{R})$: the map $(u,p) \mapsto up : \mathrm{O}(n) \times \mathcal{P}_n(\mathbb{R}) \cap \mathrm{Sym}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ is a homeomorphism.

   (b) Deduce that a general polar decomposition holds in $\mathrm{M}_n(\mathbb{R})$: each A factors as $A = uP$, $u \in \mathrm{O}(n)$, $P = P^T$ with $(Px, x) \geq 0$ for x in $\mathbb{R}^n$.

2. Deduce from polar decompostition, above, the form of polar decomposition in $\mathrm{SL}_n(\mathbb{R})$.

3. Use the fact that $\mathrm{SO}(n)$ is connected to prove that $\mathrm{SL}_n(\mathbb{R})$ is connected.

4. Let $\mathcal{U} = \{X \in \mathrm{M}_n(\mathbb{C}) : X$ has no eigenvalues in $(-\infty, -1]\}$. For $X \in \mathcal{U}$ let
$$L(X) = X \int_0^1 (I + tX)^{-1} dt.$$
   Show that $p \mapsto L(p - I) : \mathcal{P}_n(\mathbb{C}) \to \mathrm{Herm}(n)$ is continuous and the inverse of exponentiation.

5. Prove the *one-parameter subgroup theorem*: if $\gamma : \mathbb{R} \to \mathrm{GL}_n(\mathbb{F})$ is a continuous homomorphism, then there is $A \in \mathrm{M}_n(A)$ such that $\gamma(t) = \exp(tA)$. [You may take for granted that for $f \in \mathcal{C}_c^2(\mathbb{R})$, $\frac{d}{dt} \int_{-\infty}^{\infty} f(t + s) ds|_{t=0} = \int_{-\infty}^{\infty} f'(s) ds$, and that such f exists.]

6. Let G be a matrix Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$.

   (a) Prove that $\mathfrak{g}$ is a $\mathbb{R}$-vector space.

   (b) Prove that $\mathfrak{g}$ is a Lie algebra.

7. Show that $\mathrm{Lie}(\mathrm{T}_n(\mathbb{F})) = \mathfrak{t}_n(\mathbb{F}) = \{X \in \mathrm{M}_n(\mathbb{F}) : X_{ij} = 0 \text{ if } i > j\}$.

8. Show that the Lie algebra of the group
$$G_\theta = \left\{ \begin{bmatrix} e^{\theta t} \cos t & e^{\theta t} \sin t & x \\ -e^{\theta t} \sin t & e^{\theta t} \cos t & y \\ 0 & 0 & 1 \end{bmatrix} : t, x, y \in \mathbb{R} \right\}$$

is the Lie algebra with basis

$$S = \begin{bmatrix} \theta & 1 \\ -1 & \theta \end{bmatrix} = \theta I + J, \; X = E_{13} \text{ and } Y = E_{23}.$$

9. Show that for $X, Y \in \mathfrak{gl}_n(\mathbb{F})$ with $[X, Y] = 0$ that $\exp(X + Y) = \exp(X)\exp(Y)$. [You may take series convegence for granted.]

10. Show that for a connected abelian matrix Lie group G with $\mathfrak{g} = \mathrm{Lie}(G)$ that $\mathfrak{g}$ is abelian and deduce that $\exp : \mathfrak{g} \to G$ is a surjective homomorphism.

11. Prove that for a matrix Lie group G, the connected component of the identity G_0 is an open and closed subgroup. Illustrate this with an example of a matrix group G for which $G_0 \neq G$.

12. Show that if $\varphi : G \to H$ is a continuous homomorphism of matrix Lie groups, then there is a Lie algebra homomorphism $d\varphi : \mathfrak{g} = \mathrm{Lie}(G) \to \mathfrak{h} = \mathrm{Lie}(H)$ for which $\exp(td\varphi(X)) = \varphi(\exp(tX))$ for $X \in \mathfrak{g}$ and $t \in \mathbb{R}$.

13. Show that if G is a connected matrix Lie group and $\pi : G \to \mathrm{GL}(\mathcal{V})$ is a finite dimensional representation, then $\mathcal{W} \leq \mathcal{V}$ is a π-invariant subspace if and only if it is a $d\pi$-invariant subspace.

14. Show for a subgroup $N \leq G$ (G connected) that $N_0 \lhd G$ if and only if $\mathfrak{n} = \mathrm{Lie}(N)$ is a Lie ideal in $\mathfrak{g} = \mathrm{Lie}(G)$.

15. Show, *Engel's theorem*: a matrix Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}X$ is nilpotent in $\mathcal{L}(\mathfrak{g})$ for each $X \in \mathfrak{g}$.

    [You maytake for granted the fact that any Lie algebra in $\mathfrak{gl}_n(\mathbb{F})$ consisting of nilpotent matrices is necessarily a nilpotent Lie algebra.]

16. **(i)** Show that if G is a matrix Lie group then $\mathrm{Lie}(G') \supset [\mathfrak{g}, \mathfrak{g}]$ where $\mathfrak{g} = \mathrm{Lie}(G)$. Deduce that if G is a solvable group, then $\mathfrak{g}$ is a solvable Lie algebra.

    (ii) Show that any representation $\pi : G \to \mathrm{GL}(\mathcal{V})$ of a connected solvable matrix Lie group, where $\mathcal{V}$ is a finite dimensional complex vector space, is simultaneously upper-triangularizable.

    [You may take Lie's theorem for granted.]

17. Show that a finite dimensional non-abelian metablian Lie algebra $\mathfrak{m}$ (i.e. $[\mathfrak{m}, \mathfrak{m}]$ is abelian and non-zero) cotains at least one of the following three subalgebras

$$\begin{aligned} \mathfrak{f} &= \operatorname{span}_{\mathbb{R}}\{X, Y\},\ [X, Y] = Y \\ \mathfrak{g}_\theta &= \operatorname{span}_{\mathbb{R}}\{X, Y_1, Y_2\},\ [X, Y_1] = \theta Y_1 - Y_2,\ [X, Y_2] = Y_1 + \theta Y_2, \\ &\qquad [Y_1, Y_2] = 0 \text{ (where } \theta \in \mathbb{R}) \\ \mathfrak{h} &= \operatorname{span}_{\mathbb{R}}\{X, Y, Z\},\ [X, Y] = Z,\ [X, Z] = 0 = [Y, Z] \end{aligned}$$

18. **(a)** Prove that $\mathfrak{sl}_2(\mathbb{F})$ is a simple Lie algebra.

    (b) Deduce that $\mathfrak{su}(2)$ is a simple Lie algebra.

    [Note that $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$.]

19. Show that if $\mathfrak{i}$ is a Lie ideal of a Lie algebra $\mathfrak{g}$, then $[\mathfrak{i}, \mathfrak{g}] = \operatorname{span}\{[X, Y] : X \in \mathfrak{i}, Y \in \mathfrak{g}\}$ is also a Lie ideal in $\mathfrak{g}$.

20. Show that if $\mathfrak{i}$ is a Lie ideal of a Lie algebra $\mathfrak{g}$, then the Killing forms $B_{\mathfrak{g}}$ on $\mathfrak{g}$ and $B_{\mathfrak{i}}$ on $\mathfrak{i}$ satisfy $B_{\mathfrak{g}}|_{\mathfrak{i}} = B_{\mathfrak{i}}$.

21. **(a)** Show that the Killing form B of a matrix Lie algebra $\mathfrak{g}$ is ad-invariant: $B(\operatorname{ad}X(Y), Z) = B(X, \operatorname{ad}X(Z))$.

    (b) Deduce that if $\mathfrak{i}$ is Lie ideal of $\mathfrak{g}$ then $\mathfrak{i}^B = \{X \in \mathfrak{g} : B(X, Y) = 0 \text{ for all } Y \text{ in } \mathfrak{i}\}$ is also an ideal.

    (c) Deduce that if $\mathfrak{g}$ is semisimple then $\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{i}^B$.

22. Let $\mathfrak{g}$ be a matrix Lie algebra. Prove that TFAE:

    (i) $\mathfrak{g}$ is semi-simple;

    (ii) $\mathfrak{g}$ admits no solvable ideals;

    (iii) the Killing form B is non-degenerate.

    [You may take Cartan's criterion for granted.]

23. Let $\mathfrak{g}$ be a matrix Lie algebra. Prove that TFAE:

    (i) $\mathfrak{g}$ is reductive: $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ with $\mathfrak{a}$ abelian, $\mathfrak{s}$ semisimple and $[\mathfrak{a}, \mathfrak{s}] = \{0\}$;

    (ii) every abelian ideal of $\mathfrak{g}$ Lie in the centre $\mathfrak{z} = \mathrm{Z}(\mathfrak{g})$ and $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{z} = \{0\}$; and

    (iii) $\operatorname{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is completely reducible.

24. Compute the Killing form B on each of

$$\mathfrak{so}(3) \quad \text{and} \quad \mathfrak{g} = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

and indicate how this form tells us whether these algebras are semi-simple or not.

25. Derive, using the left invariant form $\eta \in \mathrm{Alt}^d(G)$, the following integral formulas for left invariant integrals:

(a) $\displaystyle\int_{\mathrm{GL}_n(\mathbb{R})} f\,|\eta| = \int_{\mathrm{GL}_n(\mathbb{R})} f([g_{ij}]) \frac{1}{|\det g|^n} \prod_{i,j=1}^{n} dg_{ij}$, $\Delta(g) = 1$. Why is $\Delta(g) = 1$ for each g in G?

(b) $$\int_{\mathrm{T}_n^0(\mathbb{R})} f(g)dg = \int_{\mathbb{R}^{(n-1)n/2}} f\left(\begin{bmatrix} 1 & x_{12} & \dots & x_{1n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & 1 & x_{n-1,n} \\ 0 & \dots & 0 & 1 \end{bmatrix} \right) \prod_{1 \le i < j \le n} dx_{ij}.$$

Why is $\Delta(g) = 1$ for each g in G?

(c) For $G = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a > 0, b \in \mathbb{R} \right\}$

$$\int_G f(g)dg = \int_{-\infty}^{\infty} \int_0^{\infty} f\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right) \frac{1}{a^2} da\, db, \quad \Delta\left(\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \right) = \frac{1}{a}.$$

(d) Let $G = G_0$ denote the Euclidean motion group of q. **??**. Then

$$\int_G f(g)dg = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{2\pi} f\left(\begin{bmatrix} \cos\theta & \sin\theta & x \\ -\sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \right) d\theta\, dx\, dy.$$

[It is sufficient to use a chart which covers all but a content zero subset of the group.]

Of course, in each case above, $f \in \mathcal{C}_c(G)$.

26. Prove *Schur's Lemma*: Let G be a matrix lie group.

(a) If $\pi : G \to \mathrm{GL}(\mathcal{V})$ and $\pi' : G \to \mathrm{GL}(\mathcal{V}')$ are finite dimensional irreducible representation then any operator $A : \mathcal{V} \to \mathcal{V}'$ which satisfies $A\pi(\cdot) = \pi'(\cdot)A$ is either invertible or zero.

(b) If $\pi : G \to \mathrm{GL}(\mathcal{V})$ is an irreducible representation on a finite dimensional $\mathbb{C}$-vector space, then the only operators $A \in \mathcal{L}(\mathcal{V})$ which satisfy $A\pi(\cdot) = \pi(\cdot)A$ are scalar multiples of I.

27. Prove *Mashcke's theorem*: for any finite dimensional representation of a compact matrix group $\pi : G \to \mathrm{GL}(\mathcal{V})$, there is an inner product $(\cdot, \cdot)$ on $\mathcal{V}$ for which π is unitary. Deduce that any π-invariant subspace $\mathcal{W} \subset \mathcal{V}$ admits a π-invariant complement.

28. Let G be a compact matrix group.

    (a) Show that there is a real inner product $(\cdot, \cdot)$ on $\mathfrak{g}$ for which

    $$([X, Y], Z) = -(Y, [X, Z]) \text{ for } X, Y, Z \in \mathfrak{g};$$

    in other words, for which each $\mathrm{ad}X$ is skew-symmtric. Deduce that $\mathfrak{m} = \mathfrak{z}^\perp$ is a Lie ideal in $\mathfrak{g}$, where $\mathfrak{z} = \mathrm{Z}(\mathfrak{g})$.

    (b) Show that for $X \in \mathfrak{g}$ that $\mathrm{ad}X$ has purely imaginary eigenvalues, and deduce that the Killing form B on $\mathfrak{g}$ is negative semi-definite, i.e. $B(X, X) \leq 0$ for $X \in \mathfrak{g}$.

    (c) Show that $[\mathfrak{g}, \mathfrak{g}]$ is semi-simple and $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$. Illustrate this with the Lie algerba $\mathfrak{u}(n)$.

29. Prove the following version of the *Peter-Weyl theorem* for a compact matrix group G

    (a) The algebra $\mathcal{M}(G)$ generated by matrix coefficients is uniformly dense in $\mathcal{C}(G)$.

    (b) For $f \in \mathcal{M}(G)$, $f(g) = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{Tr}\left(\hat{f}(\pi)\pi(g)\right)$, where $\hat{f}(\pi) = \int_G f(g)\pi(g^{-1})dg$, and $\|f\|_2{}^2 = \sum_{\pi \in \widehat{G}} d_\pi \|\hat{f}(\pi)\|_2{}^2$.

30. Show that for a compact matrix group G the set of characters $\{\chi_\pi : \pi \in \widehat{G}\}$ is an orthonormal basis for the space of central matrix coefficient functions $\mathcal{ZM}(G)$.

31. Use Schur functions to prove the *Weyl dimension formula* for irreducible representations of $\mathrm{U}(n)$: if $\lambda \in \mathbb{Z}^n_+$, then

    $$d_\lambda = \frac{\prod_{1 \leq i < j \leq n}(\lambda_i - \lambda_j - i + j)}{\prod_{1 \leq i < j \leq n}(j - i)}.$$

Deduce that the only one-dimensional representations of $\mathrm{U}(n)$ are $g \mapsto (\det g)^k$ where $k \in \mathbb{Z}$.

32. **(a)** Show that each irreducible representation $\pi : \mathrm{U}(n) \to \mathrm{U}(\mathcal{V}_\pi)$ restricts to an irreducible representation of $\mathrm{SU}(n)$.

    (b) Show that every irreducible representation of $\mathrm{SU}(n)$ is the restriction of an irreducible representation of $\mathrm{U}(n)$.

33. **(a)** Show that if G is a compact matrix group, and $\pi, \sigma \in \widehat{G}$ with associated characters χ_π, χ_σ, then $\chi_\pi \chi_\sigma = \sum_{\tau \in \widehat{G}} m_{\tau,\pi\otimes\sigma} \chi_\tau$ where $m_{\tau,\pi\otimes\sigma}$ are non-negative integers $m_{\tau,\pi\otimes\sigma}$. What is the meaning of these integers?

    (b) Give an explict parameterisation of $\widehat{\mathrm{U}}(2)$ and use this, and the question above, to gain one of $\widehat{\mathrm{SU}}(2)$.

    (d) Show that $\widehat{\mathrm{SO}}(3)$ may be identified with those elements of $\widehat{\mathrm{SU}}(2)$ which factor through $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{U}(\mathfrak{sl}_2(\mathbb{C}))$ (usual inner product on matrices).

    (d) Use (a), (b) and (c) to calculate the decomposition of the tensor products of any two representations of $\mathrm{SO}(3)$ into irreducible subrepresentations.

34. **(a)** Show that the standard representation $\iota : \mathrm{U}(n) \to \mathrm{U}(n)$ is irreducible and corresponds to the dominant weight $(1, 0, \ldots, 0)$.

    (b) Show that the complexified adjoint representaion $\mathrm{Ad} : \mathrm{U}(n) \to \mathfrak{gl}_n(\mathbb{C})$ is equivalent to $1 \oplus \pi_{(1,0,\ldots,0,-1)}$.

    [Use (a) and Schur's lemma to show that $\mathfrak{sl}_n(\mathbb{C})$ is irreducible for Ad. Show that E_{1n} is a higest weight vector.]