Orthonormal bases in Euclidean spaces

A *Euclidean space* is a $\mathbb{C}$-vector space equipped with an inner pr0duct $(\cdot,\cdot)$. We let $\|f\|_2 = (f,f)^{1/2}$ denote the norm associated to the inner proct. [Thus we take for granted the Cauchy-Schwarz inequality which shows that this is, indeed, a norm.] If $(\mathcal{E}, \|\cdot\|_2)$ is complete, we call $\mathcal{E}$ a *Hilbert space.* The othonormal basis theorem, below, needs no requirements of completeness.

A set $\{e_\alpha\}_{\alpha\in A} \subseteq \mathcal{E}$ is called *orthonormal* if $(e_\alpha, e_\beta) = \delta_{\alpha,\beta}$ (Kroenecker delta) for α, β in A. Let us recall *Pythagoreas' identity*, that if $\{e_1, \ldots, e_n\}$ is a finite orthonormal set then

$$\left\| \sum_{i=1}^n a_i e_i \right\|_2^2 = \sum_{i=1}^n |a_i|^2$$

for $a_1, \ldots, a_n$ in $\mathbb{C}$. Let us also note a handy optimisation result.

Lemma. *Let $a_1, \ldots, a_n$ be a fixed finite sequence of complex numbers. Then*

$$\inf \left\{ \sum_{i=1}^n |c_i|^2 - 2\sum_{i=1}^n \mathrm{Re}[a_i\overline{c_i}] : c_1, \ldots, c_n \in \mathbb{C} \right\} = -\sum_{i=1}^n |a_i|^2.$$

Furthermore, this infemum is acheived at $(c_1, \ldots, c_n) = (a_1, \ldots, a_n)$.

Proof. Recall the standard inequility of real numbers

$$2ac \leq a^2 + c^2 \tag{$\dagger$}$$

which is immediate from the observation that $(a-c)^2 \geq 0$. We then compute

$$\begin{aligned} \sum_{i=1}^n |c_i|^2 - 2\sum_{i=1}^n \mathrm{Re}[a_i\overline{c_i}] &\geq \sum_{i=1}^n |c_i|^2 - 2\sum_{i=1}^n |a_i||c_i| \\ &\geq \sum_{i=1}^n |c_i|^2 - \sum_{i=1}^n \left(|a_i|^2 + |c_i|^2\right), \text{ by } (\dagger) \\ &= -\sum_{i=1}^n |a_i|^2. \end{aligned}$$

Furthermore, the choice $(c_1, \ldots, c_n) = (a_1, \ldots, a_n)$ shows that the above lower bound is acheived, whence the minimum, *a fortiori* the infemum. □

Orthonormal Basis Theorem. *Let $\mathcal{E}$ be a Euclidean space and $\{e_\alpha\}_{\alpha\in A}$ be an orthonormal set. Then the following are equivalent:*

(i) $\mathrm{span}\{e_\alpha\}_{\alpha\in A}$ *is dense in $\mathcal{E}$;*

(ii) *for each f in $\mathcal{E}$ we have*

$$f = \sum_{\alpha\in A}(f, e_\alpha)e_\alpha$$

i.e. for any $\epsilon > 0$, there is a finite $F \subseteq A$ such that

$$\left\| f - \sum_{\alpha\in F'}(f, e_\alpha)e_\alpha \right\|_2 < \epsilon, \text{ for each finite } F' \supseteq F;$$

(iii) *for each f in $\mathcal{E}$ we have*

$$\|f\|_2{}^2 \;=\; \sum_{\alpha\in A}|(f, e_\alpha)|^2 \;:=\; \sup_{\substack{F\subseteq A\\ \text{finite}}}\sum_{\alpha\in F}|(f, e_\alpha)|^2.$$

We will call such a set $\{e_\alpha\}_{\alpha\in A}$, as above, an *orthonormal basis.*

Proof. First, fix finite $F = \{\alpha_1, \ldots, \alpha_n\} \subseteq A$ and let $\mathcal{E}_F = \mathrm{span}\{e_\alpha\}_{\alpha\in F}$. If $c_1, \ldots, c_n \in \mathbb{C}$, a straightforward computation, using Pythagoreas' identity yields

$$\begin{aligned}\left\| f - \sum_{i=1}^n c_i e_{\alpha_i} \right\|_2^2 &= \left(f - \sum_{i=1}^n c_i e_{\alpha_i}\,,\; f - \sum_{i=1}^n c_i e_{\alpha_i} \right) \\ &= \|f\|_2{}^2 - 2\sum_{i=1}^n \mathrm{Re}[(f, e_{\alpha_i})\overline{c_i}] + \sum_{i=1}^n |c_i|^2.\end{aligned}$$

Then, the lemma above shows that

$$\begin{aligned}\mathrm{dist}(f, \mathcal{E}_F)^2 &= \inf\left\{ \left\| f - \sum_{i=1}^n c_i e_{\alpha_i} \right\|_2^2 : c_1, \ldots, c_n \in \mathbb{C} \right\} \\ &= \|f\|_2{}^2 - \sum_{i=1}^n |(f, e_{\alpha_i})|^2 = \left\| f - \sum_{i=1}^n (f, e_{\alpha_i})e_{\alpha_i} \right\|_2^2.\end{aligned}$$

Hence we see that

$$\operatorname{dist}(f,\mathcal{E}_F)^2 = \|f\|_2{}^2 - \sum_{\alpha\in F} |(f,e_\alpha)|^2 \tag{††}$$

$$= \left\| f - \sum_{\alpha\in F} (f,e_\alpha)e_\alpha \right\|_2^2. \tag{‡}$$

We now observe that $\operatorname{dist}(f,\mathcal{E}_F) \geq \operatorname{dist}(f,\mathcal{E}_{F'})$ if $F \subseteq F'$ and that $\operatorname{span}\{e_\alpha\}_{\alpha\in A} = \bigcup_{F\subseteq A \text{ finite}} \operatorname{span}\mathcal{E}_F$ by definition of linear span. In particular, if (i), holds if and only if for any f in $\mathcal{E}$

$$0 = \operatorname{dist}(f,\mathcal{E}) = \inf\{\operatorname{dist}(f,\mathcal{E}_F) : F \subseteq \mathcal{E} \text{ is finite}\}. \tag{$\diamondsuit$}$$

However, (‡) shows that ($\diamondsuit$) is equivalent to (ii). Meanwhile, (††), and the observation that $\sum_{\alpha\in F} |(f,e_\alpha)|^2 \leq \sum_{\alpha\in F'} |(f,e_\alpha)|^2$ if $F \subseteq F'$, show that ($\diamondsuit$) is equivalent to (iii). □

We remark, for sake of context, the following. Its proof is left to the interested reader.

Riesz-Fischer Theorem. *Let $\mathcal{E}$ be a Euclidean space with an orthonormal basis $\{e_\alpha\}_{\alpha\in A}$. Then $\mathcal{E}$ is a Hilbert space if and only if*

$$\sum_{\alpha\in A} c_\alpha e_\alpha := \lim_{\substack{F\nearrow A \\ F\subset A \text{ finite}}} \sum_{\alpha\in F} c_\alpha e_\alpha \text{ converges in } \mathcal{E},$$

$$\text{whenever } (c_\alpha)_{\alpha\in A} \subset \mathbb{C} \text{ satisfies } \sum_{\alpha\in A} |c_\alpha|^2 := \sup_{\substack{F\subset A \\ \text{finite}}} \sum_{\alpha\in F} |c_\alpha|^2 < \infty.$$

The limit, above, is to be interpreted as in part (ii) of the Orthonormal Basis Theorem.

Exercise. Consider the Euclidean space $\mathcal{C}[0,1]$ of continuous functions on $[0,1]$ with inner product $(f,g) = \int_0^1 f\bar{g}$ (Riemann integral). Show that $\{t \mapsto e^{i2\pi nt}\}_{n\in\mathbb{Z}}$ is an orthonormal basis in $\mathcal{C}[0,1]$. However, $\sum_{n\in\mathbb{Z}} \frac{i}{2\pi n}[1-(-1)^n]e^{i2\pi nt}$ does not converge in $\mathcal{C}[0,1]$, hence this Euclidean space is not complete. [Hint: $\frac{i}{2\pi n}[1-(-1)^n] = \int_0^{1/2} e^{i2\pi nt}\,dt$.]

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