PMATH 753
Candidate Final Exam Questions

$\{\text{questions on final}\} \subset \{\text{questions on this sheet}\}$

A Give a clear statment of the following theorem:

1. Hahn-Banach Theorem
2. Banach-Steinhaus Theorem
3. Open Mapping Theorem
4. Inverse Mapping Theorem
5. Closed Graph Theorem
6. Separation Theorem
7. Tychonff's Theorem
8. Alaoglu's Theorem
9. w*-Separation Theorem
10. Goldstine's Theorem
11. Krein-Milman Theorem
12. Orthogonal Complementation Theorem
13. Riesz Representation Theorem (for Hilbert spaces)
14. Riesz-Fischer Theorem
15. Orthonormal Basis Theorem (incl. Bessel's ineq. and Parseval id.)
16. Inversion Theorems 1 & 2.
17. Spectral Radius Formula
18. Spectral Theorem (for a compact operator on a Banach space)

19. Spectral Theorem (for a compact hermitian [or normal] operator on a Hilbert space)

B Provide a proof of the following. You should be prepared to provide a definition of any term in *italics*.

1. Prove that $\boldsymbol{c}_0{}^* \cong \ell_1$ and $\ell_1{}^* \cong \ell_\infty$. Also show that $\boldsymbol{c}_0$ is *weak**-dense (i.e. $\sigma(\ell_\infty, \widehat{\ell_1})$-dense) in ℓ_∞.

2. Prove that if $\mathcal{X}$ is an infinite dimensional normed vector space, then $B(\mathcal{X})$ is not compact in the $\|\cdot\|$-topology. [Use A1 to show that if $\mathcal{Y}$ is a proper closed subspace of $\mathcal{X}$, $0 < \varepsilon < 1$, then there is $x \in B(\mathcal{X})$ with $\operatorname{dist}(x, \mathcal{Y}) > 1 - \varepsilon$.]

3. If $\mathcal{X}$ is a Banach space, then it cannot have a countable infinite *Hamel basis*. [Use Baire.]

4. Prove the Inverse Mapping Theorem. [Use A3.]

5. Prove the Closed Graph Theorem. [Use A4.]

6. If $(\mathcal{X}, \|\cdot\|)$ is a Banach space, and $|||\cdot|||$ is another norm which dominates $\|\cdot\|$ and for which $(\mathcal{X}, |||\cdot|||)$ is complete, then $|||\cdot|||$ is equivalent to $\|\cdot\|$.

7. Prove Goldstine's Theorem. [Hint: use A8 and A6.]

8. Let $\mathcal{U}$ be an *ultrafilter* on $\mathbb{N}$. If $x \in \ell_\infty$ prove that the ultrafilter limit $\lim_{\mathcal{U}} x = \lim_{n \in U \in \mathcal{U}} x_n$ exists. [You might want to recall what $\ell_\infty{}^*$ looks like; alternatively, consider the net $(x_n)_{(n,U)\in N} \subset \|x\|_\infty \mathbb{B}$ where $N = \{(n, U) : n \in U \in \mathcal{U}\}$ subject to an appropriate preordering, and use q. 20.]

9. Show that a Banach limit exists on ℓ_∞, i.e. a norm one functional $L \in \ell_\infty{}^*$ such that $L(Tx) = L(x)$ for each x where $T : \mathcal{B}(\ell_\infty)$ is the back-shift operator, $T(x_1, x_2, \dots) = (x_2, x_3, \dots)$. [Either use a certain H.B.T. argument as in A2; or consider $\lim_{\mathcal{U}} \circ M$ where M is a Cesaro mean operator.]

10. Show that if $\mathcal{X}$ is a Banach space for which $B(\mathcal{X})$ is weakly compact, then $\mathcal{X}$ is *reflexive*. Deduce that if $\mathcal{X}^*$ is reflexive then $\mathcal{X}$ is reflexive too. [Use A10 and A8.]

11. Show that if $\mathcal{X}$ is reflexive, and $\mathcal{Y} \subset \mathcal{X}$ is a closed subspace, then $\mathcal{Y}$ is reflexive too.

12. Show that a separable reflexive Banach space has separable dual.

13. Show that in a *Hausdorff space* (X, τ), that any *compact* subset is closed.

14. Show that if $(X, \tau), (Y, \sigma)$ are compact topological spaces with (Y, σ) Hausdorff, then a continuous bijection $\varphi : X \to Y$ is a *homeomorphism*.

15. Show that the following are homeomorphic: (i) $\{z \in \ell^\infty_{\mathbb{R}} : |z| = 1\}$ (w* topology), (ii) $\{-1, 1\}^{\mathbb{N}}$ (product topology), and (iii) the Cantor set C (metric/order topology).

16. Prove that $(B(\mathcal{X}^*), w^*)$ is *metrisable* if and only if $\mathcal{X}$ is separable.

17. Prove that in an infinite dimensional normed vector space that $\overline{S(\mathcal{X})}^w = B(\mathcal{X})$.

18. Prove that in a finite dimensional normed vector space that $\tau_{\|\cdot\|} = \sigma(\mathcal{X}, \mathcal{X}^*)$.

19. Prove that any *net* has an *ultranet* as a *subnet*.

20. Prove that a topological space is *compact* if and only if every net has a *cluster point* if and only if every ultranet *converges*.

21. Prove Tychonoff's Theorem. [Use either f.i.p., ultrafilters or ultranets.]

22. Prove Alaoglu's theorem. [Use either A8, or ultranets.]

23. Compute the set of *extreme points* $\text{ext}B(\ell^1)$, and show that, $\overline{\text{conv}}^{\|\cdot\|_1}\text{ext}B(\ell^1) = B(\ell^1)$.

24. Show that if C is a complete convex set in a Euclidean space $\mathcal{X}$ and $x \in \mathcal{X}$, then there is a unique y in C for which $\text{dist}(x, C) = \|x - y\|$. [Hint: Parallelogram Law as in proof of A12.]

25. In a Euclidean space, prove that $(x, y) = 1$, for $\|x\| = \|y\| = 1$, if and only if $x = y$. [Hint: examine proof of Cauchy-Schwarz.]

    Thus if $\mathcal{E}$ is a Euclidean space compute $\text{ext}B(\mathcal{E})$. Show that $B(\mathcal{E}) = \text{conv}\,\text{ext}B(\mathcal{E})$, even though $\mathcal{E}$ may not be complete.

26. Show that none of $\boldsymbol{c}_0, \mathcal{C}^{\mathbb{R}}[0,1]$ nor $L_1[0,1]$ can be dual spaces. [Hint: Show that $\text{ext}B(\mathcal{X})$ is too small.]

27. Prove that in a Banach space, if $(x_n)_{n=1}^{\infty}$ is a sequence which converges to x_0, then $\overline{\text{co}}\{x_n\}_{n=1}^{\infty}$ is a compact set.

28. Prove that any *separable* Euclidean space admits an orthogonal sequence whose span is dense in the space. [Gram-Schmidt.]

29. Prove the Riesz-Fischer theorem which characterises Hilbert spaces amongst Euclidean spaces. [It's a good exercise to prove Bessel's inequlity, while you're at it.]

30. Consider $L_2[0,1]$ with *orthonormal basis* $(e_n)_{n=-\infty}^{\infty}$ where $e_n(t) = e^{i2\pi nt}$. Let D in $\mathcal{L}(\text{span}\{e_n\}_{n=-\infty}^{\infty})$ denote the operator of differentiation. Let $\mathcal{D} = \{f \in L_2[0,1] : \sum_{n=-\infty}^{\infty} |n^2(f, e_n)|^2 < \infty\}$. Show that there is a unique extension of D^2 from $\mathcal{D}$ to $L_2[0,1]$.

    Moreover, $D^2 : \mathcal{D} \to L_2[0,1]$ has *closed graph* in $L_2[0,1] \oplus L_2[0,1]$.

31. Show that a sequence $\{x_n\}_{n=1}^{\infty}$, in a Hilbert space $\mathcal{H}$ with orthonormal basis $(e_i)_{i \in I}$, converges weakly to $x_0 \Leftrightarrow$ each sequence $\{(x_n, e_i)\}_{n=1}^{\infty}$ converges to (x_0, e_i) and $\sup_{n \in \mathbb{N}} \|x_n\| < +\infty$.

    Deduce that any orthonormal sequence $\{f_n\}_{n=1}^{\infty}$ converges weakly to 0.

32. **(a)** Let $\mathcal{H}$ be a separable Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Show that T in $\mathcal{L}(\mathcal{H})$ (linear operators from $\mathcal{H}$ to $\mathcal{H}$) is bounded $\Leftrightarrow$ there is an infinite matrix $[t_{ij}]$ such that for $x \in \mathcal{H}$ and $i \in \mathbb{N}$, $(Tx, y) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} t_{ij}(x, e_j)\overline{(y, e_i)}$. Moreover, $t_{ij} = (Te_j, e_i)$ for each pair i, j. [Use A15 and method from Assign. 2 q.6.]

    (b) In the question above, what does the matrix for T^* look like?

33. Compute the *spectrum*, *point spectrum* and *approximate point spectrum* for any of the following $\mathbb{C}$-linear operators:

    (a) an *idempotent* E in $\mathcal{B}(\mathcal{X})$, $\mathcal{X}$ a Banach space.

    (b) a multiplcation operator M_a in $\mathcal{B}(\ell_p)$, $a \in \ell_\infty$, $1 \leq p < \infty$.

    (c) the unilatral shift operator S in $\mathcal{B}(\ell_2)$, $S(x_1, x_2, \dots) = (0, x_2, x_2, \dots)$.

34. Let $\mathcal{X}$ be an infinite dimensional complex Banach space.

(a) Prove that if K in $\mathcal{B}(\mathcal{X})$ is compact, and we have sequence $(\alpha_n)_{n=1}^{\infty} \subset \mathbb{C}$ and subspaces $\mathcal{Y}_0 \subsetneq \mathcal{Y}_1 \subsetneq \mathcal{Y}_2 \subsetneq \dots$ such that $(\alpha_n I - K)\mathcal{Y}_n \subseteq \mathcal{Y}_{n-1}$, then $\lim_{n\to\infty} \alpha_n = 0$.

(b) Prove that if K is compact then $\sigma_p(K)$, if it is infinite, is a sequence converging to 0. Deduce that $\sigma(K) = \sigma_p(K) \cup \{0\}$. [I.e. prove part of A18. You may use the facts that each generalised eigenspace for a non-zero eignevalue is finite dimensional; that in $\mathbb{C}$ a compact set with countable boundary is itself countable; that eigenspaces corresponding to disticnt eigenvalues are linearly independant; and that the boundary of the spectrum is in the approximate point spectrum.]

(c) Show, by way of example, that for compact K we may have $\sigma_p(K) = \varnothing$.

35. Show that T in $\mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ complex Hilbert space) is *normal* if and only if $\|Tx\| = \|T^*x\|$ for each x in $\mathcal{H}$.

36. **(a)** Show that for T in $\mathcal{B}(\mathcal{H})$, $\|T^*T\| = \|T\|^2$.

(b) Show that that $\|T^n\| = \|T\|^n$ if T is normal.

37. **(a)** Show that $\overline{\mathcal{F}(\ell_p)}^{\|\cdot\|} = \mathcal{K}(\ell_p)$ $(1 \leq p < \infty)$, i.e. that the *finite rank* bounded operators on ℓ_p are norm dense in the compact operators. [Let $P_n \in \mathcal{B}(\ell_p)$ be the usual projection. Show that $\|P_nK - K\| \to 0$. Use a technique from the proof (given in class) of the fact that K^* is compact.]

(b) Determine which multiplication operators M_a in $\mathcal{B}(\ell_p)$, $a \in \ell_\infty$, are compact.

38. Show that a *separable* normed vector space may be isometrically embedded into $\mathcal{B}(\ell_p)$ $(1 \leq p \leq \infty)$. [Hint: first isometrically embed $\mathcal{X}$ into ℓ_∞, then use multiplication operators from ℓ_∞.]

39. Prove the following theorem of Schauder. For a Banach space $\mathcal{X}$ and $K \in \mathcal{B}(\mathcal{X})$, that $K \in \mathcal{K}(\mathcal{X})$ if and only if $K^* \in \mathcal{K}(\mathcal{X}^*)$. [For necessity, use either the proof from class, or the Arzela-Ascoli proof posted on the website.]

40. **(a)** Show that if $ST = TS$ in $\mathcal{B}(\mathcal{X})$, then $\exp S \exp T = \exp(S+T)$.

(b) Show that if H is a *hermitian* operator on a Hilbert space $\mathcal{H}$, then $\exp(iH)$ defines a unitary operator.

(c) Prove the following theorem of Fuglede. If S in $\mathcal{B}(\mathcal{H})$ is *normal*, and T in $\mathcal{B}(\mathcal{H})$ commutes with S, then T commutes with S^*.

41. **(a)** Prove that if $H \in \mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ a Hilbert space) is hermitian and $\lambda \in \sigma_p(H)$, then the orthogonal projection $P = P_{\ker(\lambda I - H)}$ satisfies $TP = PT$ whenever $TH = HT$ for T in $\mathcal{B}(\mathcal{H})$.

(b) Deduce that if $H = H^*$ in $\mathcal{K}(\mathcal{H})$ with $\ker H = \{0\}$, and $TH = HT$, then there is a sequence of mutually orthogonal finite rank projections $(P_k)_{k=1,2,\dots}$ such that $Tx = \sum_{k=1,2,\dots} P_k T P_k x$ for each x in $\mathcal{H}$. [Use A19.]

42. Let $k \in \mathcal{C}([0,1]^2)$, $K : L_2[0,1] \to L_2[0,1]$ be given by $Kf(x) = \int_0^1 k(x,y)f(y)dy$ (Lebesgue integral). Prove that $K \in \mathcal{K}(L_2[0,1])$. Moreover, show that K is hermitian if $\overline{k(x,y)} = k(y,x)$ for x, y in $[0,1]^2$.