PMATH 753, FALL 2012

Assignment #5 Due: December 3

1. If $\mathcal{X}$ is a Banach space and E in $\mathcal{B}(\mathcal{X})$ is an idempotent, i.e. $E^2 = E$, compute the point spectrum $\sigma_p(E)$ and the spectrum $\sigma(E)$.

2. **(a)** Show that if $\mathcal{X}$ is a Banach space and $S, T \in \mathcal{B}(\mathcal{X})$, then

$$\sigma(ST) \cup \{0\} = \sigma(TS) \cup \{0\}.$$

[Hint: verify $(zI - TS)^{-1} = z^{-1}I + z^{-1}T(zI - ST)^{-1}S$ for appropriate $z \in \mathbb{C}$.]

(b) Show, by way of example, that $\sigma(ST)$ may differ from $\sigma(TS)$.

3. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces.

(a) $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is called *adjointable* if $T^*f \in \mathcal{X}^*$ for each $f \in \mathcal{Y}^*$. Prove that T is adjointable only if it is bounded.

[Hint: Banach-Steinhaus.]

(b) Show that $S \in \mathcal{B}(\mathcal{Y}^*, \mathcal{X}^*)$ is $\sigma(\mathcal{Y}^*, \mathcal{Y})$-$\sigma(\mathcal{X}^*, \mathcal{X})$ continuous if and only if $S = T^*$ for some T in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

[Hint: F in $\mathcal{X}^{**}$ is $\sigma(\mathcal{X}^*, \widehat{\mathcal{X}})$-continuous $\Leftrightarrow$ $F = \hat{x}$ (see proof of w^*-Separation Theorem) $\Leftrightarrow \lim_\nu F(f_\nu) = F(f_0)$ whenever $f_0 = \sigma(\mathcal{X}^*, \mathcal{X})$-$\lim_\nu f_\nu$ A3, Q5.]

4. Let $a = (a_n)_{n=1}^\infty$ in ℓ_∞. Let the *multiplication operator* $M_a : \ell_p \to \ell_p$ $(1 \leq p \leq \infty)$ be given by $M_a(x_n)_{n=1}^\infty = (a_n x_n)_{n=1}^\infty$.

(a) Compute $\|M_a\|$, $\sigma_p(M_a)$, $\sigma_{ap}(M_a)$ and $\sigma(M_a)$.

[Hint: show that $\sigma_{ap}(T) \supseteq \overline{\sigma_p(T)}$.]

(b) If K is a compact subset of $\mathbb{C}$, show that there is an operator $T \in \mathcal{B}(\ell_p)$ for which $\sigma(T) = K$.

(c) Give a characterisation of those a in ℓ_∞ for which M_a is a compact operator.

5. Let $f \in \mathcal{C}[0,1] = \mathcal{C}^{\mathbb{C}}[0,1]$ with $f \neq 0$. Let $M_f : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ be given by $M_f g = fg$.

   (a) Compute $\|M_f\|$ and $\sigma(M_f)$.

   (b) Characterise those f in $\mathcal{C}[0,1] \setminus \{0\}$ for which $\sigma_p(M_f) \neq 0$.

   (c) Show that M_f, as above, is never a compact operator.

6. Let $\mathcal{X}$ be a Banach space and $T, S \in \mathcal{B}(\mathcal{X})$.

   (a) Show that the series $\exp T := \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} T^n$ converges in norm.

   (b) Show that if $TS = ST$, then $\exp S \exp T = \exp(S + T)$.

   (c) Show that if $\mathcal{H}$ is a Hilbert space, and H in $\mathcal{B}(\mathcal{H})$ satisfies $H^* = H$, then $\exp(iH)$ is a unitary operator.

7. Let $\mathcal{H}$ be a Hilbert space and $R, S, T \in \mathcal{B}(\mathcal{H})$ be so $R^*R = RR^*$, $T^*T = TT^*$ and $RS = ST$. Prove that $R^*S = ST^*$.

   [Hint: expand $\exp(R^* - R)S \exp(T - T^*)$ to show that $\|\exp(R^*)S \exp(-T^*)\| \leq \|S\|$; then apply Liouville's Theorem (complex analysis) to the entire function $F_f(z) = f\big(\exp(zR^*)S \exp(-zT^*)\big)$ for any $f \in \mathcal{B}(\mathcal{H})^*$ to deduce that $\exp(zR^*)S = S \exp(zT^*)$.]

8. Let $\mathcal{H}$ be a Hilbert space and P in $\mathcal{B}(\mathcal{H})$ be an idempotent, i.e. $P^2 = P$. Show that the following are equivalent:

   (i) P is an orthogonal projection

   (ii) P is self-adjoint: $P^* = P$

   (iii) P is normal: $P^*P = PP^*$

   (iv) $(Px, x) = \|Px\|^2$ for every x in $\mathcal{H}$.