PMATH 753, FALL 2012

Assignment #4 Due: November 14

1. Compute the set of *extreme points* of the unit ball, $\mathrm{ext}B(\mathcal{X})$, for the following $\mathbb{R}$-Banach spaces $\mathcal{X}$, i.e. don't worry about the complex cases.

   (a) $\mathcal{X} = L_1[0,1]$ [Hint: show that $F(s) = \int_{[0,s]} |f|$ is continuous.]

   (b) $\mathcal{X} = \ell_1$

   (c) $\mathcal{X} = \ell_\infty$

   (d) Show that $\mathrm{co}\,\mathrm{ext}B(\ell_1)$ is $\|\cdot\|_1$-dense in $B(\ell_1)$.

2. Let $B_\mathbb{R} = B\left(\mathcal{C}^\mathbb{R}[0,1]\right)$ and $B_\mathbb{C} = B\left(\mathcal{C}^\mathbb{C}[0,1]\right)$.

   (a) Show that $\mathrm{ext}B_\mathbb{R} = \{-\mathbf{1}, \mathbf{1}\}$ where $\mathbf{1}$ is the constant function with range 1.

   (b) Deduce that $\mathcal{C}^\mathbb{R}[0,1]$ is not a dual space.

   (c) Show that $\mathrm{ext}B_\mathbb{C} = \mathcal{C}^\mathbb{S}[0,1] := \{f \in \mathcal{C}^\mathbb{C}[0,1] : |f(t)| = 1 \text{ for each } t \text{ in } [0,1]\}$.

   (d) Show that $\mathrm{co}\,\mathcal{C}^\mathbb{S}[0,1]$ is $\|\cdot\|_\infty$-dense in $B_\mathbb{C}$.

   [Hint: if $z \in \mathbb{C}$, $0 < |z| \leq 1$ then $z = \frac{1}{2}\mathrm{sgn}z(|z| + i\sqrt{1-|z|^2} + |z| - i\sqrt{1-|z|^2})$ – draw a picture with $|z|$, first, to see how one gets this. Thus if $f \in B_\mathbb{C}$, $|f(t)| > 0$ for all t, we can show that $f \in \mathrm{co}\,\mathcal{C}^\mathbb{T}[0,1]$.]

   (e) (Bonus) Is $\mathcal{C}^\mathbb{C}[0,1]$ a dual space?

3. A sequence $(x_n)_{n=1}^\infty$ in a Banach space $\mathcal{X}$ is called *null* if $\lim_{n\to\infty} x_n = 0$ (norm limit).

   (a) Show that if $(x_n)_{n=1}^\infty$ is a null sequence, then its norm closed convex hull $C = \overline{\mathrm{co}}\{x_n\}_{n=1}^\infty$ is compact.

   [From PMath 351, it suffices to show that C is *totally bounded*.]

   (b) Show that a closed set K in a Banach space $\mathcal{X}$ is compact, if and only if there is a null sequence $\{x_n\}_{n=1}^\infty$ such that $K \subset \overline{\mathrm{co}}\{x_n\}_{n=1}^\infty$. Deduce that the closed convex hull of a compact set in a Banach space is compact.

   [Note that $2K$ is compact; cover with closed 1/2-balls, $y_i + \frac{1}{2}B(\mathcal{X})$. Do this again with $K_1 = \bigcup_{i=1}^n([2K \cap (y_i + \frac{1}{2}B(\mathcal{X}))] - y_i)$. Continue.]

4. Show that in a non-complete Euclidean space, the decomposition theorem $\mathcal{E} = \mathcal{F} \oplus \mathcal{F}^{\perp}$ can fail for a closed subspace $\mathcal{F}$.

   [Hint: consider your favorite non-complete Euclidean space $\mathcal{E}$, let $\mathcal{H}$ be its completion, and $\mathcal{F} = \{f_0\}^{\perp} \cap \mathcal{E}$ for a judicious choice of f_0 in $\mathcal{H} \setminus \mathcal{E}$.]

5. **(a)** Let $\mathcal{H}$ be a Hilbert space and $P \in \mathcal{B}(\mathcal{H})$ satisfy $P^2 = P$ and $\|P\| = 1$. Show that P is the orthogonal projection onto $\mathcal{M} = \mathrm{ran}P$.

   [Hint: one must show that $(\ker P)^{\perp} = \mathcal{M}$. First establish that $(\ker P)^{\perp} \subset \mathcal{M}$; note that $\ker P = \mathrm{ran}(I - P)$.]

   (b) Exhibit a Hilbert space $\mathcal{H}$ and a projection $P = P^2$ in $\mathcal{B}(\mathcal{H})$ for which $\|P\| > 1$.

6. Let $q(t) = t^2 - 1$ and define the family of (non-normalised) *Legendre polynomials* by $p_n = D^n[q^n]$ where D is the differentiation operator.

   (a) Show that $(p_n)_{n=0}^{\infty}$ is an orthogonal sequence in $L_2[-1,1]$.

   [Hint: odd $\perp$ even, use integration by parts]

   (b) Deduce that, up to normalisation, this family is the family acheived by applying Gram-Schmdit orthogonalisation to the sequence of basic monomials $(m_n)_{n=0}^{\infty}$, $m_n(t) = t^n$; and, moreover, that $(p_n)_{n=0}^{\infty}$ is a *complete* orthogonal set, i.e. its linear span is dense.

   (c) Show that $D[q\, Dp_n]$ is orthogonal to each m_k where $k < n$. Deduce that p_n solves the nth *Legendre equation*: $(t^2 - 1)p''(t) + 2tp'(t) = n(n+1)p(t)$.

   (d) Let $\mathcal{D} = \{f \in L_2[-1,1] : \sum_{n=1}^{\infty} |n(n+1)(f, \tilde{p}_n)|^2 < \infty\}$ where each $\tilde{p}_n = \frac{1}{\|p_n\|_2} p_n$. Show that for each f in $\mathcal{D}$, the expression $L(f) = q\, D^2 f + 2m_1\, Df$ defines an element in $L_2[-1,1]$. Moreover, when $\mathcal{D}$ is endowed with the norm from $L_2[-1,1]$ then $L : \mathcal{D} \to L_2[-1,1]$ has closed graph.

   [Hint: first, find a new inner product on $\mathcal{D}$ for which $\mathrm{span}\{p_n\}_{n=1}^{\infty}$ is dense in $\mathcal{D}$ and which allows L to be viewed as a bounded operator.]

   REMARK. A densly defined operator with closed graph os often called a *closed* operator on a Hilbert space. This are used in physical applications.