PMATH 753, FALL 2012

Assignment #3 Due: October 31

1. Let $\mathcal{X}$ be a normed vector space. Let $B(\mathcal{X}) = \{x \in \mathcal{X} : \|x\| \leq 1\}$, $S(\mathcal{X}) = \{x \in \mathcal{X} : \|x\| = 1\}$ and $w = \sigma(\mathcal{X}, \mathcal{X}^*)$ denote the weak topology. Show that:

   (a) $\dim \mathcal{X} = \infty \quad \Rightarrow \quad \overline{S(\mathcal{X})}^w = B(\mathcal{X})$.

   [Hint: Begin by showing that every weak neighbourhood of 0 contains a subspace.]

   (b) $\dim \mathcal{X} < \infty \quad \Leftrightarrow \quad w = \tau_{\|\cdot\|}$.

2. Let $1 < p < \infty$.

   (a) Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in ℓ_p and $x^{(0)} \in \ell_p$. Show that

$$w\text{-}\lim_{n\to\infty} x^{(n)} = x^{(0)} \quad \Leftrightarrow \quad \begin{array}{c} \lim_{n\to\infty} x_i^{(n)} = x_i^{(0)} \text{ for each } i \\ \text{and } \sup_{n\in\mathbb{N}} \left\|x^{(n)}\right\|_p < +\infty. \end{array}$$

   (b) Must the same happen for nets?

3. Show that the following sets are pairwise *homeomorphic*, i.e. there exists a continuous open bijection between any pair of them.

   (i) $\big(\{\chi_E : E \in \mathcal{P}(\mathbb{N})\}, w^*\big)$, $w^* = \sigma(\ell_\infty, \widehat{\ell_1})$, relativised to this set.

   (ii) $\big(\{0,1\}^{\mathbb{N}}, \pi\big)$, π is the product topology where each component $\{0,1\}$ has discrete topology.

   (iii) (C, τ), C is "middle thirds" Cantor set in $[0,1]$, τ is the usual metric/order topology.

   [Hint: elements of C are profitably expressed in *ternary* form, as opposed to decimal.]

4. Show that if $B(\mathcal{X}^*)$, in the weak* topology, is metrisable – i.e. there exists a metric on $B(\mathcal{X}^*)$ which gives the relativised weak* topology – then $\mathcal{X}$ must be separable.

   [Hint: If the weak* topology is metrisable, then there must be a countable sequence of neighbourhoods with intersection $\{0\}$.]

The next 2 questions address the sufficiency of nets, in general topology.

5. Let (X, τ) and (Y, σ) be toplogical spaces and $\varphi : X \to Y$ be a function. Show that

$$\varphi \text{ is } \tau\text{-}\sigma\text{-continuous} \quad \Leftrightarrow \quad \begin{array}{c} \sigma\text{-}\lim_{\nu \in N} \varphi(x_\nu) = \varphi(x_0) \\ \text{whenever } \tau\text{-}\lim_{\nu \in N} x_\nu = x_0 \end{array}$$

for any net $(x_\nu)_{\nu \in N}$ and any point x_0, in X.

6. **(a)** Let (X, τ) be a topological space. Show that the following are equivalent
   (i) (X, τ) is compact;
   (ii) every net in X admits a τ-cluster point; and
   (iii) every ultranet in X admits a τ-limit point.
   [Hint: f.i.p. is the key to understanding compactness.]
   (b) Use ultranets to give an alternative proof to Tychonff's theorem.

The next 2 questions address the necessity of nets.

7. A topological space (X, τ) is called *sequentially compact* if for every sequence in X admits a τ-converging subsequence. Compact metric spaces are sequentially compact from PMATH 351.

   (a) Let $I = [0, 1]$ and let $X = I^I$ be endowed with the product topology π. Show that (X, π) is not sequentially compact.
   [Hint: try $x_t^{(n)} = 3^n t - \lfloor 3^n t \rfloor$.]
   (b) Why does this not contradict Q6 (a)?
   (c) (Bonus) Exhibit a cluster point of the sequence you used to prove part (a), above.

8. **(a)** Prove that if $(x^{(n)})_{n=1}^\infty$ is a sequence in ℓ_1, which converges weakly to $x^{(0)}$ in ℓ_1, then $\lim_{n\to\infty} \left\| x^{(n)} - x^{(0)} \right\|_1 = 0$.
   [Hint: If $\sigma(\ell^1, \widehat{\boldsymbol{c}_0})$-$\lim_{n\to\infty} x^{(n)} = 0$ but $\left\|x^{(n)}\right\|_1 \geq \varepsilon > 0$, then $(x^{(n)})_{n=1}^\infty$ cannot converge weakly to 0.]

   No sequence in $S(\ell_1)$ converges to 0; by Q1 (a), some net in $S(\ell_1)$ must.

   (b) (Bonus) Are weakly compact subsets of ℓ_1 norm compact?