PMATH 753, FALL 2012

Assignment #2 Due: October 8

If $\mathcal{X}$ is a vector space over $\mathbb{F}$, a *Hamel basis* is any subset B which is:

• *linearly independant:* every finite subset of B is linearly independant;

• *spanning:* span$B$, the space of finite linear combinations of elements from B, is all of $\mathcal{X}$.

1. Show that an infinite dimensional Banach space does not admit a countable Hamel basis. [Hint: Baire.]

2. Let $1 \leq p < \infty$.

   (a) Show that $|\ell_p| = \mathfrak{c}$, i.e. the cardinality of ℓ_p is that of the continuum.

   (b) Show that there exists a family $\mathcal{F}$ of subsets of $\mathbb{N}$ with the following properties:
   (i) if $E, F \in \mathcal{F}$ then $E \cap F$ is finite or empty,
   (ii) $|\mathcal{F}| = \mathfrak{c}$.
   [Hint: the solution seems irrational.]

   (c) Show, without using the continuum hypothesis, that ℓ_p admits a Hamel basis of cardinality $\mathfrak{c}$.

   (d) Show that any Hamel basis for ℓ_p must have cardinality $\mathfrak{c}$.

   Some Banach spaces $\mathcal{X}$, such as ℓ_p, admit a different type of basis called a *Schauder basis*: a sequence $(e_n)_{n=1}^{\infty}$ of elements such that for each x in $\mathcal{X}$, there is a unique sequence of scalars $(x_n)_{n=1}^{\infty}$ for which $x = \lim_{n\to\infty} \sum_{i=1}^{\infty} x_i e_i$. Results (a), (c) and (d) will all hold for such $\mathcal{X}$.

3. (Bonus) Given a Schauder basis $\{e_n\}_{n=1}^{\infty}$, show that the functional $|||x||| = \sup_{n\in\mathbb{N}} \|\sum_{i=1}^{n} x_i e_i\|$, where $(x_n)_{n=1}^{\infty}$ is the sequence of coefficients above, defines a norm on $\mathcal{X}$ which is equivalent to $\|\cdot\|$. Hence each projection defined by $P_n x = \sum_{i=1}^{n} x_i e_i$ is bounded, and the sequence of these projections is uniformly bounded.

   [No bonus marks will be given for the trivial parts of this.]

4. Let $\ell_\infty = \ell_\infty^{\mathbb{R}}$, the Banach space of bounded real sequences with uniform norm $\|\cdot\|_\infty$. This exercise describes Banach's *generalised limits* of bounded sequences.

   (a) Show that if $\mathcal{Y}$ is any subspace of ℓ_∞, then the functional $p : \ell_\infty \to \mathbb{R}$, $p(x) = \operatorname{dist}(x, \mathcal{Y})$, is sublinear, with $p(x) \leq \|x\|_\infty$ for every x.

   (b) Show that there exists a linear functional $L : \ell_\infty \to \mathbb{R}$ such that

      (i) $\|L\| = 1$ and $L(\mathbf{1}) = 1,$ where $\mathbf{1} = (1, 1, \dots),$ and

      (ii) $L(n{*}x) = L(x),$ where $n{*}x = (x_{n+1}, x_{n+2}, \dots)$ if $x \in \ell_\infty, n \in \mathbb{N}$.

   [Hint: $\mathcal{Y} = \operatorname{span}\{x - 1{*}x : x \in \ell_\infty\}$; $\operatorname{dist}(\mathbf{1}, \mathcal{Y}) = ?$]

   (c) Show that $\liminf\limits_{n\to\infty} x_n \leq L(x) \leq \limsup\limits_{n\to\infty} x_n$.

   [Hint: first show that $L|_{\boldsymbol{c}_0} = 0$; then $L(x) \geq 0$ if $x_n \geq 0$.]

   (d) Fix $m \in \mathbb{N}$ and let $x_n = n/m - \lfloor n/m \rfloor$, where $\lfloor s \rfloor = \max\{k \in \mathbb{N} : k \leq s\}$ for s in $\mathbb{R}$. Compute $L(x)$. [Hint: this sequence is periodic.]

5. **(a)** Show that if $\mathcal{Y}$ is a subspace of a normed vector space $\mathcal{X}$, then its closure, the smallest closed subset containing $\mathcal{Y}$, may be realised as
$$\overline{\mathcal{Y}} = \bigcap \{\ker f : f \in \mathcal{X}^* \text{ and } \mathcal{Y} \subset \ker f\}.$$

   (b) Show that if $\mathcal{X}^*$ is *separable*, i.e., there is a countable dense subset in $\mathcal{X}^*$, then $\mathcal{X}$ must be separable too.

   (c) Is it true that if $\mathcal{X}$ is separable, then $\mathcal{X}^*$ is also separable? Prove your assertion.

6. Let $\mathcal{X}$ be an $\mathbb{F}$-vector space, and $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ each be norms under which $\mathcal{X}$ is complete.

   (a) Show that $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ are either *equivalent*, i.e. there are $m, M > 0$ such that

   $$m\,\|x\| \leq |\!|\!|x|\!|\!| \leq M\,\|x\| \quad \text{for each } x \text{ in } \mathcal{X}$$

   or their topologies are incomparable, i.e.

   $$\tau_{\|\cdot\|} \not\subset \tau_{|\!|\!|\cdot|\!|\!|} \text{ and } \tau_{|\!|\!|\cdot|\!|\!|} \not\subset \tau_{\|\cdot\|}. \qquad (\clubsuit)$$

   (b) Is it possible for ($\clubsuit$) to occur, with complete norms?

7. Let $1 < p, r < \infty$ (we do not suppose these to be conjugate). Suppose $A = [a_{ij}]_{i,j=1}^{\infty}$ is an infinite matrix with the property that for each $x = (x_1, x_2, \dots)$ in ℓ_p, each series $\sum_{j=1}^{\infty} a_{ij}x_j$ converges for each i on $\mathbb{N}$, and

   $$Ax = \left(\sum_{j=1}^{\infty} a_{1j}x_j, \sum_{j=1}^{\infty} a_{2j}x_j, \dots\right) \in \ell_r.$$

   Show that A, i.e. $x \mapsto Ax$, defines a bounded operator from ℓ_p to ℓ_r.

   [Hint: you might want to warm-up by checking that the rows of A must be ℓ_q-sequences, $\frac{1}{p} + \frac{1}{q} = 1$.]

8. Let

   $$\mathcal{C}^1[0,1] = \left\{ f \in \mathcal{C}[0,1] : \begin{array}{c} f \text{ is differentiable on } (0,1), \\ \text{right differentiable at } 0 \\ \text{and left differentiable at } 1, \\ \text{with } f' \in \mathcal{C}[0,1] \end{array} \right\}$$

   and let this space be equiped with the uniform norm $\|\cdot\|_\infty$.

   (a) Show that the differentiation operator $D : \mathcal{C}^1[0,1] \to \mathcal{C}[0,1]$, $Df = f'$, is unbounded but has a closed graph. [Hint: F.T. of C.]

   (b) Is D an open map?