PMATH 753, FALL 2012

Assignment #1 Due: September 21

1. Let (X, σ) and (Y, τ) be topological spaces and $f : X \to Y$ be a function. For any subset E of Y we let $f^{-1}(E) = \{x \in X : f(x) \in E\}$; thus $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ is a set function.

   As with metric topology, we define a subset E of X to be σ-closed, or simply *closed*, if $X \setminus E \in \sigma$, i.e., its compliment is open.

   Show that the following are equivalent:

   (i) f is σ-τ continuous at every point in X.

   (ii) $f^{-1}(V)$ is open for each open subset V of Y.

   (iii) $f^{-1}(F)$ is closed for each closed subset F of Y.

2. Let $p, q > 1$ be so $\frac{1}{p} + \frac{1}{q} = 1$; $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Exercises (a) and (b) show that there is a linear isometric isomorphism: ${\ell_p}^* \cong \ell_q$.

   (a) Show that if $b = (b_1, b_2, \dots) \in \ell_q$, then the map $f_b : \ell_p \to \mathbb{F}$ given by
   $$f_b\big((x_1, x_2, \dots)\big) = \sum_{k=1}^{\infty} x_k b_k$$
   defines a linear functional which is bounded with $\|f_b\| = \|b\|_q$.

   (b) Show that every bounded linear functional on ℓ_p arises as in (a).

   (c) Deduce that $(\ell_p, \|\cdot\|_p)$ is complete.

   (d) Determine which sequence space (if any) describes ${\ell_1}^*$. Prove your assertion.

3. Let $\boldsymbol{c}(\mathbb{Z}) = \{x \in \ell_\infty(\mathbb{Z}) : \lim_{n\to+\infty} x_n \text{ and } \lim_{n\to-\infty} x_n \text{ each exist}\}$.

   (a) Let $L_+, L_- : \boldsymbol{c}(\mathbb{Z}) \to \mathbb{F}$ be given by $L_+(x) = \lim_{n\to+\infty} x_n$, $L_- = \lim_{n\to-\infty} x_n$. Show that $L_-, L_+ \in \boldsymbol{c}(\mathbb{Z})^*$.

   (b) Show that for each $y = (y_{-\infty}, \dots, y_{-1}, y_0, y_1, \dots, y_\infty) \in \ell_1(\mathbb{Z} \cup \{-\infty, +\infty\})$ the map $f_y : \boldsymbol{c}(\mathbb{Z}) \to \mathbb{F}$,
   $$f_y(x) = y_{-\infty} L_-(x) + \sum_{i\in\mathbb{Z}} y_i x_i + y_\infty L_+(x)$$
   is a bounded linear functional with $\|f_y\| = \|y\|_1$.

(c) Show that each element of $\boldsymbol{c}(\mathbb{Z})^*$ arises as in (b).

(d) (Bonus question) Is there an isometric linear bijection $T : \boldsymbol{c}(\mathbb{Z}) \to \boldsymbol{c}_0$?

4. This exercise is concerned with computing the dual of the space of the bounded sequences, ℓ_∞.

(a) If $E \subset \mathbb{N}$, let χ_E denote the sequence with $\chi_{E,i} = 1$ if $i \in E$, and $\chi_{E,i} = 0$ otherwise. Show that the space of simple sequences, $\mathcal{S} = \mathrm{span}\{\chi_E : E \subset \mathbb{N}\}$, is dense in ℓ_∞.
[For any set $\mathcal{E}$ in a vector space $\mathcal{X}$, we let span$\mathcal{E}$ denote the smallest subspace which contains $\mathcal{E}$.]

(b) Let $\mathcal{FA}(\mathbb{N})$ denote the space of all functions $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{F}$ which satisfy
(i) *finite additivity*: $\mu(E \cup F) = \mu(E) + \mu(F)$ if $E \cap F = \varnothing$, and
(ii) *bounded variation*:

$$V(\mu) = \sup \left\{ \sum_{j=1}^{n} |\mu(E_j)| : \mathbb{N} = E_1 \dot{\cup} \ldots \dot{\cup} E_n \right\} < +\infty$$

where $\dot{\cup}$ denotes disjoint union.
Show that each μ in $\mathcal{FA}(\mathbb{N})$ determines a well-defined linear functional $f_\mu : \mathcal{S} \to \mathbb{F}$ by

$$f_\mu(x) = \sum_{j=1}^{n} \alpha_j \mu(E_j) \quad \text{whenever} \quad \begin{array}{l} x = \sum_{j=1}^{n} \alpha_j \chi_{E_j} \text{ and} \\ E_j \cap E_k = \varnothing \text{ if } j \neq k. \end{array}$$

Moreover, this functional is bounded on the normed vector space $(\mathcal{S}, \|\cdot\|_\infty)$ with $\|f_\mu\| = V(\mu)$.

(e) Hence deduce that f_μ extends to a unique bounded linear functional on ℓ_∞.

(d) Show that every bounded linear functional on ℓ_∞ arises as above. Hence there is a linear isometric isomorphism: ${\ell_\infty}^* \cong \mathcal{FA}(\mathbb{N})$, where $\mathcal{FA}(\mathbb{N})$ has pointwise operations and norm $\|\mu\| = V(\mu)$.

(e) (Bonus question) Is every element of $\mathcal{FA}(\mathbb{N})$ of the form $\mu(E) = \sum_{j \in E} \mu_j$ with $\sum_{j=1}^{\infty} |\mu_j| < +\infty$? You must prove your assertion.