PMATH 753

On ultrafilters and their role in Tychonoff's Theorem

Let $X \neq \varnothing$. We say a family of subset $\mathcal{U} \subset \mathcal{P}(X)$ is an *ultrafilter* if

- $\mathcal{U}$ has finite intersection property (f.i.p); and
- for any $A \subset X$, either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Notice that for $A \subset X$, $A \in \mathcal{U}$ if and only if $A \cap U \neq \varnothing$ for all U in $\mathcal{U}$.

Ultrafilter Lemma. *Any family $\mathcal{F} \subset \mathcal{P}(X)$ with f.i.p. is contained in an ultrafilter.*

Proof. Let $\Phi = \{\mathcal{G} \in \mathcal{P}(X) : \mathcal{F} \subset \mathcal{G}$ and $\mathcal{G}$ has f.i.p.$\}$. We assign a partial order by inclusion. If $\Gamma \subset \Phi$ is a chain, then let $\mathcal{G}_\Gamma = \bigcup_{\mathcal{G} \in \Gamma} \mathcal{G}$. Trivially $\mathcal{F} \subset \mathcal{G}$. If $G_1, \dots, G_n \in \mathcal{G}_\Gamma$ then $G_i \in \mathcal{G}_i$ for each i, and, up to reindexing $\mathcal{G}_1 \subset \cdots \subset \mathcal{G}_n$, so $G_1, \dots, G_n \in \mathcal{G}_n$. Hence $\bigcap_{i=1}^n G_i \neq \varnothing$ as $\mathcal{G}_n \in \Phi$. Hence $\mathcal{G}_\Gamma \in \Phi$, and is clearly an upper bound for Γ. Hence by Zorn's Lemma, a maximal element $\mathcal{U}$ for Φ exists.

For a finite sequence $U_1, \dots, U_n$ in $\mathcal{U}$, let $A = \bigcap_{i=1}^n U_i$. We see that $\mathcal{U} \cup \{A\}$, satisfies f.i.p., so is an element of Φ. Hence $\mathcal{U} \cup \{A\} = \mathcal{U}$, by maximaility, i.e. $A \in \mathcal{U}$. (This is the part I forgot, in class.) Now, if $A \subset X$ and we have that $A \cap U \neq \varnothing$ for each U in $\mathcal{U}$ then $\mathcal{U} \cup \{A\} \in \Phi$, since $\mathcal{U}$ is closed under finite intersection. Hence again, $A \in \mathcal{U}$. If $A \cap U = \varnothing$ for at least one U in $\mathcal{U}$, then it is easy to see that $\mathcal{U} \cup \{X \setminus A\} \in \Phi$, hence as before, $X \setminus A \in \mathcal{U}$. □

Tychonoff's Theorem. *If $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ is a family of compact spaces, then $X = \prod_{\alpha \in A} X_\alpha$ is comapct in propduct topology π.*

Proof. Let $\mathcal{F} \in \mathcal{P}(X)$ be any family with f.i.p. Let $\mathcal{U} \supset \mathcal{F}$ be an ultrafilter, which exists by the Ultrafilter Lemma.

Fix α in A. We observe that for $U_1, \dots, U_n \in \mathcal{U}$ that $\bigcap_{i=1}^n p_\alpha(U_i) \supset p_\alpha\left(\bigcap_{i=1}^n U_i\right) \neq \varnothing$. (Notice, A. of C. just got used.) Hence, by compactness, there is $x_\alpha \in \bigcap_{U \in \mathcal{U}} \overline{p_\alpha(U)}^{\tau_\alpha}$. Hence for V with $x_\alpha \in V \in \tau_\alpha$, we have $V \cap p_\alpha(U) \neq \varnothing$ for each U in $\mathcal{U}$, and it follows that $p_\alpha^{-1}(V) \in \mathcal{U}$.

We apply the result above to all indices α to get a point $x = (x_\alpha)_{\alpha \in A}$ in X. As above, for each α and each V with $x_\alpha \in V \in \tau_\alpha$ we have $p_\alpha^{-1}(V) \in \mathcal{U}$. Hence each basic π-open nbhd. of x, being a finite interesction of sets $p_\alpha^{-1}(V)$, is in $\mathcal{U}$. Thus $x \in \overline{U}^\pi$ for each U in $\mathcal{U}$, so $x \in \bigcap_{U \in \mathcal{U}} \overline{U}^\pi \subset \bigcap_{F \in \mathcal{F}} \overline{F}^\pi$. Hence (X, π) is compact. □

Proposition. *If we assume Tychonoff's Theorem, then the Axiom of Choice is true.*

This is pretty weird, considering that the general form Tychonoff seems to require non-empty products, but bear with me.

Proof. Let $\{S_\alpha\}_{\alpha \in A}$ be a family of sets. Find a point p which is not in $\bigcup_{\alpha \in A} S_\alpha$. Let $X_\alpha = S_\alpha \cup \{p\}$ with topology $\tau_\alpha = \{\varnothing, S_\alpha, \{p\}, X_\alpha\}$. Then (X_α, τ_α) is compact as τ_α is finite. Hence by Tychonoff's Theorem $X = \prod_{\alpha \in A} X_\alpha$ is compact with product topology. Observe that $X \neq \varnothing$, since it contains points $(x_\alpha)_{\alpha \in A}$ where $x_\alpha = p$ for all but finitely many indices α. (Naively speaking, finitary selection is always allowed.) Let $U_\alpha = p_\alpha^{-1}(\{p\}) \in \pi$, for each α. Then no finite collection $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a cover of X, hence by compactness $\{U_\alpha\}_{\alpha \in A}$ cannot cover X. Thus $\prod_{\alpha \in A} S_\alpha = X \setminus \bigcup_{\alpha \in A} U_\alpha \neq \varnothing$. $\square$