PMATH 753

Compactness of adjoint operators via Arzela-Ascoli

Theorem. *If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and K is a compact linear operator form $\mathcal{X}$ to $\mathcal{Y}$, then $K^* : \mathcal{Y}^* \to \mathcal{X}^*$ is also compact*

Proof. Let $\Omega = \overline{K(B(\mathcal{X}))} \subset \mathcal{Y}$, which is a compact metric space. Let $R : \mathcal{Y}^* \to \mathcal{C}(\Omega)$ be given by $Rf = f|_\Omega$. Observe that if $f \in \mathcal{Y}^*$ then the density of $K(B(\mathcal{X}))$ in Ω provides

$$\|Rf\|_\infty = \sup_{y \in \Omega} |f(y)| = \sup_{x \in B(\mathcal{X})} |f(Kx)| = \sup_{x \in B(\mathcal{X})} |K^* f(x)| = \|K^* f\|.$$

Hence $Rf \mapsto K^* f : R(\mathcal{Y}^*) \to \mathcal{X}^*$ is an isometry. Thus if $R(B(\mathcal{Y}^*))$ is totally bounded in $\mathcal{C}(\Omega)$, then $K^*(B(\mathcal{Y}^*))$ is totally bounded in $\mathcal{X}^*$, which means that K^* is indeed compact.

First note that for f in $B(\mathcal{Y}^*)$ that $\|Rf\| = \|K^* f\| \leq \|K^*\| = \|K\|$, so $R(B(\mathcal{Y}^*))$ is equi-bounded. Next, if $y, y' \in \Omega$ then for f in $B(\mathcal{Y}^*)$ we have

$$|Rf(y) - Rf(y')| \leq \|Rf\| \, \|y - y'\| \leq \|K\| \, \|y - y'\|$$

which means that $R(B(\mathcal{Y}^*))$ is equi-Lipschitz, in particular equi-continuous. Thus by Arzela-Ascoli Theorem, $R(B(\mathcal{Y}^*))$ is totally bounded. □

Remark. This proof has the advantage over the proof given in class, that it does not rely on Alaoglu's theorem, and hence on Axiom of Choice. However, the proof in class shows that $K^* : \mathcal{Y}^* \to \mathcal{X}^*$ is w^*-norm continuous on bounded sets, and hence $K^*(B(\mathcal{Y}^*))$ is itself norm-closed in $B(\mathcal{X}^*)$. This is slightly finer information than the proof above can immediatley supply. However, if we return the the proof that $\mathcal{B}(\mathcal{Y}^*, \mathcal{X}^*)$ is complete we see that what we showed was that $\mathcal{B}(\mathcal{Y}^*, \mathcal{X}^*)$ could be identified as a closed subspace of the complete space $\mathcal{C}^{\mathcal{Y}^*}(B(\mathcal{X}^*))$, where $B(\mathcal{X}^*)$ is understood as a metric space with norm topology. The same technique proof will show that $R(B(\mathcal{Y}^*))$ is norm closed in $\mathcal{C}(\Omega)$; hence $K^*(B(\mathcal{Y}^*))$ is norm-closed in $\mathcal{X}^*$.