PMATH 753

A non-Baire proof of Banach-Steinhaus theorem

This uses a so-called "gliding-hump" technique. It is weaker than the Baire-based proof since the other one shows that an unbounded family of operators can only be pointwise bounded on a meager set of points, wheras this proof reveals only that some sequence may be constructed on which an unbounded family of operators is unbounded at some point.

Banach-Steinhaus Theorem. *Let $\mathcal{X}$ be Banach space and $\mathcal{Y}$ be a normed space and $\mathcal{F} \subset \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then if*

$$\sup\{\|Tx\| : T \in \mathcal{F}\} < \infty \text{ for all } x \text{ in } \mathcal{X}$$

we must have that

$$\sup\{\|T\| : T \in \mathcal{F}\} < \infty.$$

Proof. (Adapted from *A Short Course in Banach Space Theory*, by N.L. Carothers.) Suppose that $\mathcal{F}$ is not uniformly bounded, i.e. $\sup_{T\in\mathcal{F}} \|T\| = \infty$. We wish to establish the existence of a point at which $\mathcal{F}$ is not bounded. We let $\mathcal{X}_0 = \{x \in \mathcal{X} : \sup_{T\in\mathcal{F}} \|Tx\| < \infty\}$. It is obvious that $\mathcal{X}_0$ is a subspace of $\mathcal{X}$. Our goal is to show that $\mathcal{X}_0 \subsetneq \mathcal{X}$, which will be amply realised if $\overline{\mathcal{X}_0} \subsetneq \mathcal{X}$. Hence we may as well assume $\mathcal{X}_0$ is dense in $\mathcal{X}$, and will see that it cannot be closed in $\mathcal{X}$.

Fix $0 < \delta < \frac{1}{2}$. Select T_1 from $\mathcal{F}$ Let x_1 in $\mathcal{X}_0$ be so $\|x_1\| = \delta$ and $\|T_1 x_1\| > (1-\delta)\|T_1\|\|x_1\|$. We now conduct an induction. Having selected $T_1, \ldots, T_{n-1}$ and $x_1, \ldots, x_{n-1}$, select T_n from $\mathcal{F}$ for which

$$\|T_n\| > \frac{M_{n-1} + n}{(1-2\delta)\delta^n}, \quad \text{where} \quad M_{n-1} = \sup_{T\in\mathcal{F}} \|T(x_1 + \cdots + x_{n-1})\|$$

and then choose x_n in $\mathcal{X}_0$ with

$$\|x_n\| = \delta^n \quad \text{and} \quad \|T_n x_n\| > (1-\delta)\|T_n\|\|x_n\| = (1-\delta)\delta^n \|T_n\|.$$

Notice that the series $\sum_{k=1}^{\infty} x_k$ has Cauchy sequence of partial sums, hence converges in the Banach space $\mathcal{X}$. Observe that the choices of T_n and x_n entail that

$$\left(1 - \frac{\delta}{1-\delta}\right)\|T_n x_n\| = \frac{1-2\delta}{1-\delta}\|T_n x_n\| > (1-2\delta)\delta^n \|T_n\| > M_{n-1} + n$$

while

$$\left\| T_n \sum_{k=n+1}^{\infty} x_k \right\| \le \|T_n\| \sum_{k=n+1}^{\infty} \delta^k = \|T_n\| \frac{\delta^{n+1}}{1-\delta} < \frac{\delta}{1-\delta} \|T_n x_n\| .$$

We put this together to compute for $x = \sum_{k=1}^{\infty} x_k$ that

$$\begin{aligned} \|T_n x\| &\ge \|T_n x_n\| - \left\| T_n \sum_{k=1}^{n-1} x_k \right\| - \left\| T_n \sum_{k=n+1}^{\infty} x_k \right\| \\ &> \left(1 - \frac{\delta}{1-\delta}\right) \|T_n x_n\| - M_{n-1} > n. \end{aligned}$$

Hence $\mathcal{F}$ is not pointwise bounded on all of $\mathcal{X}$; at best it is pointwise bounded on a dense subspace.

Notice that the point of this proof is that we may write

$$T_n x = \underbrace{T_n \sum_{k=1}^{n-1} x_k}_{\text{norm} \le M_{n-1}} + \underbrace{T_n x_n}_{\text{norm} >> M_{n-1}} + \underbrace{T_n \sum_{k=n+1}^{\infty} x_k}_{\text{norm} << \|T_n x_n\|}$$

so that the growth of $T_n x_n$ drives the growth of $T_n x$. The series defining x "humps", for T_n, at n, and is relatively tame otherwise; it uniformly sums bad phenomena for all T_n, simultaneously. In building the proof, we selected vectors x_n to be summable via a geometric series (probably primarily because these are the only sequences we really understand), and choose the growth of operators T_n, afterwards. □