PMATH 753

Axiom of Choice et al

Definition/Notation. Given any non-empty set, S, a *binary relation* R is simply a subset of the Cartesian product $S \times S$. We tend to write "$s\,R\,t$" instead of "$(s,t) \in R$".

Definition. Let S be a non-empty set. A binary relation $\leq$ on S is called a *partial ordering* if it satisfies, for s, t, u in S

(i)	$s \leq s$	(reflexivity)
(ii)	$s \leq t,\ t \leq u \Rightarrow s \leq u$	(transitivity)
(iii)	$s \leq t,\ t \leq s \Rightarrow s = t$	(antisymmetry)

We call the pair $(S, \leq)$ a *partially ordered set*. In $(S, \leq)$, a *chain* is any subset C if any two elements are comparable, i.e. for any s, t in C, either $s \leq t$ or $t \leq s$. If S is a chain in $(S, \leq)$, we say that $\leq$ is a *total ordering* on S. If A is any subset of S, an *upper bound* for A (w.r.t. $\leq$) is any u in S for which $s \leq u$ for s in A. A *well-ordering* is any ordering $\leq$ on S such that in any non-empty subset A there is a *minimal element*, i.e. a in A such that $a \leq s$ for s in A.

Observe that a well-ordered set is totally ordered.

Examples.

(i) If $X \neq \varnothing$, then $(\mathcal{P}(X), \subseteq)$ is a partially ordered set.

(ii) Let $\leq$ be the usual ordering on $\mathbb{R}$. Then $(\mathbb{R}, \leq)$ and $(\mathbb{Q}, \leq)$ are totally ordered. The set $(\mathbb{N}, \leq)$ is well-ordered, as is $(\{n - \frac{1}{k}\}_{n,k\in\mathbb{N}}, \leq)$.

Theorem. *The following statments are equivalent:*

(i) Axiom of choice: *for every non-empty X, there is a* choice function, *i.e.* $\gamma : \mathcal{P}(X) \setminus \{\varnothing\} \to X$ *such that* $\gamma(A) \in A$ *for each* A.

(ii) Hausdorff's maximality principle: *in any partially ordered set* $(S, \leq)$ *there is a* maximal chain, *i.e. a chain M for which no $M \cup \{s\}$ is a chain for any s in $S \setminus M$.*

(iii) Zorn's Lemma: *if in a partially ordered set $(S, \leq)$, each chain has an upper bound, then there is a* maximal element m for S, i.e. $m \leq s$ implies $m = s$.

(iv) Well-ordering principle: *any non-empty set S admits a well-ordering.*

Proof. (i) $\Rightarrow$ (ii). We first prove an ancilliary result, based on axiom of choice.

(I) Let $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfy

- $\varnothing \in \mathcal{F}$, and
- if $\mathcal{K} \subset \mathcal{F}$ is a chain (w.r.t $\subseteq$), then $\bigcup_{K\in\mathcal{K}} K \in \mathcal{F}$.

Then $\mathcal{F}$ contains an element M such that $M \cup \{x\} \notin \mathcal{F}$ for any $x \in X \setminus M$.

Let us prove this statment. For each A in $\mathcal{F}$ let $A^* = \{x \in X : A \cup \{x\} \in \mathcal{F}\}$. (Note that this choice is dependant on $\mathcal{F}$, but we need not acknowledge that explicitly.) We fix a choice function $\gamma : \mathcal{P}(X) \setminus \{\varnothing\} \to X$. We let $\Gamma(A) = A \cup \{\gamma(A^*)\}$ if $A^* \neq \varnothing$, and $\Gamma(A) = A$ otherwise. We note that $\gamma(A^*) \in A^*$ for each A in $\mathcal{F}$ for which $A^* \neq \varnothing$, and hence $\Gamma(A) \in \mathcal{F}$. (Observe that if there were M in $\mathcal{F}$ for which $M^* = \varnothing$ we would be done, but we prefer to leave the condtradiction aspect of this proof to the end.)

We define a *tower* (really, a $(\gamma, \mathcal{F})$-tower) to be any subcollection $\mathcal{T} \subseteq \mathcal{F}$ for which

- $\varnothing \in \mathcal{T}$,
- $A \in \mathcal{T} \Rightarrow \Gamma(A) \in \mathcal{T}$
- if $\mathcal{K} \subset \mathcal{T}$ is a chain (w.r.t $\subseteq$), then $\bigcup_{K\in\mathcal{K}} K \in \mathcal{T}$.

Notice that $\mathcal{F}$, itself, is a tower, and that the intersection of any family of towers is again a tower. Hence

$$\mathcal{T}_0 = \bigcap\{\mathcal{T} : \mathcal{T} \subseteq \mathcal{F} \text{ is a tower}\}$$

is a tower. Notice, $\varnothing \in \mathcal{T}_0$, and hence $\{\gamma(\varnothing^*)\}, \{\gamma(\varnothing^*), \gamma(\{\gamma(\varnothing^*)\}^*)\} \in \mathcal{T}_0$, etc. We aim to show that $(\mathcal{T}_0, \subseteq)$ is totally ordered. To this end, we call a set C in $\mathcal{T}_0$ *comparable* (in $\mathcal{T}_0$), if for A in $\mathcal{T}_0$, either $A \subseteq C$ or $C \subseteq A$. For such C consider the family

$$\mathcal{T}_C = \{A \in \mathcal{T}_0 : A \subsetneq C\} \cup \{C\} \cup \{A \in \mathcal{T}_0 : \Gamma(C) \subseteq A\}.$$

We observe that $\varnothing \in \mathcal{T}_C$. If $A \in \mathcal{T}_C$ then $\Gamma(A) \in \mathcal{T}_0$, and, using the assumption the C is comparable, we see that

- if $A \subsetneq C$, then $\Gamma(A) \subseteq C$, since otherwise, in the case that $A^* \neq \varnothing$, we would have $A \subsetneq C \subsetneq A \cup \{\gamma(A^*)\}$, which is clearly impossible; or
- if $A = C$ or if $\Gamma(C) \subseteq A$ then $C \subseteq A \subseteq \Gamma(A)$;

hence $\Gamma(A) \in \mathcal{T}_C$. Moreover, if $\mathcal{K}$ is a chain in $\mathcal{T}_C$, then let $B = \bigcup_{K\in\mathcal{K}} K$. Indeed if each $K \subseteq C$, then $B \subseteq C$; and if $\Gamma(C) \subseteq K$ for some K, then $\Gamma(C) \subseteq B$. Thus $\mathcal{T}_C$ is a tower, in which case we must have $\mathcal{T}_C = \mathcal{T}_0$, as $\mathcal{T}_0$ is the minimal tower in $\mathcal{F}$. It follows that $\Gamma(C)$ is comparable if C is. Thus the family of comparable sets, $\mathcal{C}$, satisfies the first two axioms of a tower; it remains to check the third. If $\mathcal{K}$ is a chain in $\mathcal{C}$, let $B = \bigcup_{K \in \mathcal{K}} K$. If $A \in \mathcal{T}_0$ then either $A \subseteq K$ for some K, in which case $A \subseteq B$; or $K \subseteq A$ for all K, in which case $B \subseteq A$. Thus $B \in \mathcal{C}$. Hence $\mathcal{C}$ is itself a tower, and again by minimality of $\mathcal{T}_0$, we see that $\mathcal{C} = \mathcal{T}_0$. Hence we have that $(\mathcal{T}_0, \subseteq)$ is indeed totally ordered, hence a chain in $(\mathcal{F}, \subseteq)$.

Now we let $M = \bigcup_{T \in \mathcal{T}_0} T \in \mathcal{T}_0$. If it were the case that $M^* \neq \varnothing$, we would have that $\Gamma(M) = M \cup \{\gamma(M^*)\} \in \mathcal{T}_0$ since $\mathcal{T}_0$ is a tower. But this violates the fact that $\gamma(M^*) \notin M$. Hence $M^* = \varnothing$ which proves (I).

(II) We now use (I) to prove (ii). Given a partially ordered set $(S, \leq)$, let $\mathcal{F}$ denote the set of all chains in S. We remark that $\varnothing$ is trivially a chain. Any chain $\mathcal{K}$ in $(\mathcal{F}, \subseteq)$ has that $C = \bigcup_{K \in \mathcal{K}} K$ is a chain, i.e. any two elements of C must live in some K. Any M, arising form the conclusion of (I), is a maximal chain.

(ii) $\Rightarrow$ (iii). Suppose $(S, \leq)$ is a partially ordered set in which each chain has a maximal element. Let M be a maximal chain in $(S, \leq)$ and m be an upper bound for M. Then $M \cup \{m\}$ is a chain, and hence equal to M by maximality of M, i.e. $m \in M$. Moreover, if any s in S satisfies $m \leq s$, then $M \cup \{s\}$ is a chain, from which it again follows that $s \in M$, hence $s \leq m$. But then $s = m$, so m is a maximal element.

(iii) $\Rightarrow$ (iv). Let $\mathcal{W} = \{(A, \leq_A) : A \in \mathcal{P}(X), \leq_A \text{ is a well-ordering on } A\}$. We let $(A, \leq_A) \leq (B, \leq_B)$ iff $(A, \leq_A)$ is an *initial segment* of $(B, \leq_B)$, i.e. $A \subseteq B$, $\leq_B |_{A \times A} = \leq_A$, and for a in A and b in B, we have $a \leq_B b$. Let $\mathcal{C}$ be a chain in $(\mathcal{W}, \leq)$. Let $U = \bigcup_{(C, \leq_C) \in \mathcal{M}} C$ and for s, t in $\mathcal{U}$, let $s \leq_U t$ whenever $s, t \in C$ with $s \leq_C t$, for some $(C, \leq_C) \in \mathcal{C}$. Then $\leq_U$ is trivially well-defined. If $A \subseteq U$ is non-empty, there is some $(C, \leq_C)$ in $\mathcal{C}$ for which $A \cap C \neq \varnothing$, and thus admits a minimal element a_C. Observe that if $A \cap C' \neq \varnothing$ for another $(C', \leq_{C'})$ in $\mathcal{C}$, then $C \subseteq C'$, say, and we see that $a_{C'} = a_C$, since $(C, \leq_C)$ is an initial segment of $(C', \leq_{C'})$. In particular, $(U, \leq_U)$ is an upper bound for $\mathcal{C}$.

Hence, by Zorn's lemma, $\mathcal{W}$ admits a maximal element $(M, \leq_M)$. If there were s in $S \backslash M$, we could let $M' = M \cup \{s\}$ and extend $\leq_M$ to M' by assigning $t \leq_{M'} s$ for all t in M. But then $(M', \leq_{M'}) \in \mathcal{W}$, which would violate the maximlity of $(M, \leq_M)$. Hence $\leq_M$ is a well-ordering on S.

(iv) $\Rightarrow$ (i). Suppose $\leq$ is a well-ordering on X. Let $\gamma(A)$ be the minimal element of A for each A in $\mathcal{P}(X) \setminus \{\varnothing\}$. $\square$

Remark. There is an equivalent formulation of axiom of choice: if $\{X_i\}_{i\in I}$ is any collection of non-empty sets, then the Cartesian product $\prod_{i\in I} X_i$ is non-empty.

Indeed, given axiom of choice, as formulated in (i), above, we let $X = \bigcup_{i\in I} X_i$. Then for any choice function $\gamma : \mathcal{P}(X) \setminus \{\varnothing\} \to X$ we note that $(\gamma(X_i))_{i\in I} \in \prod_{i\in I} X_i$.

Conversely, if X is any non-empty set, let we suppose $\prod_{A\in\mathcal{P}(X)\setminus\{\varnothing\}} A \neq \varnothing$, i.e. contains an element $(x_A)_{A\in\mathcal{P}(X)\setminus\{\varnothing\}}$. Then $\gamma(A) = x_A$ defines a choice function.

Remark. Let us finally remark that finite Cartesian products of non-empty sets, $\prod_{i=1}^{n} X_i$, may be regarded as non-empty in absence of axiom of choice. Naively speaking, this is a *finitary* selction process, and not problematic.

The family of functions form B to A, which we may denote A^B (for non-empty A and B), can be generally considered non-empty without appealing to axiom of choice. In fact, one may define a function from B to A as any subset f of $B \times A$ such that for any b in B such that $(b, a), (b, a') \in f$ implies $a = a'$. (We willfully confuse a function with its graph. Sorry! Moreover, we shall prefer notations "$f(b) = a$" or "$b \mapsto f(b)$", rather than "$(b, a) \in f$".) For example, fixing a in A, $B\times\{a\}$, i.e. $b \mapsto a$, is a constant function.