PMATH 453/753

Nets

Definition/Notation. A *directed set* is a pair $(N, \leq)$ where

(ds1) $\leq$ is a *pre-order* on N, i.e. it is *symmetric*: $\nu \leq \nu$, and *transitive*: $\nu \leq \nu'$ and $\nu' \leq \nu'' \Rightarrow \nu \leq \nu''$ for ν, ν', ν'' in N; and

(ds2) $\leq$ is *cofinal*: given any ν, ν' in N, there is ν'' in N so $\nu \leq \nu''$ and $\nu' \leq \nu''$.

Given a non-empty set X, a *net* in X is a function $x : N \to X$. We usually write $x_\nu = x(\nu)$ and denote the net by $(x_\nu)_{\nu \in N}$.

Given a net $(x_\nu)_{\nu \in N}$ in X an A in $\mathcal{P}(X)$, we say that $(x_\nu)_{\nu \in N}$ is

- *eventually* in A if there is ν_A n N for which $x_\nu \in A$ whenever $\nu \geq \nu_A$; and
- *frequently* in A if for any ν in N, there is ν' in N with $\nu' \geq \nu$ and $x_{\nu'} \in A$.

Suppose $(M, \leq)$ is another directed set. A *cofinal map* is any function $\varphi : M \to N$ which satisfies

(cm) for any ν in N, there is μ_ν in M such that $\varphi(\mu) \geq \nu$ whenever $\mu \geq \mu_\nu$.

A *subnet* of a net $(x_\nu)_{\nu \in N}$ is any net of the form $(x_{\varphi(\mu)})_{\mu \in M}$. We usually write $\nu_\mu = \varphi(\mu)$, and hence denote the subnet by $(x_{\nu_\mu})_{\mu \in M}$.

We call $\varphi : M \to N$ a *directed map* if it satisfies

(dm1) $\mu \leq \mu'$ in $M \Rightarrow \varphi(\mu) \leq \varphi(\mu')$ in N; and

(dm2) for any ν in N there is μ in M so $\nu \leq \varphi(\mu)$.

It is easy to see that directed maps are cofinal. Directed maps are used for subnets in the book of S. Willard, whereas cofinal maps are used in the book of J.L. Kelley.

Examples. (i) $(\mathbb{N}, \leq)$ (usual order) is a directed set, so sequences are nets. Subsequences are a special types of subnets.

(ii) Any non-empty subset of $\mathbb{R}$ is directed with usual ordering. The map $t \mapsto \lfloor t \rfloor : (1, \infty) \to \mathbb{N}$ is a cofinal map (even a directed map), and hence $(x_{\lfloor t \rfloor})_{t \in (1,\infty)}$ is a subnet of a sequence $(x_n)_{n=1}^{\infty}$.

(iii) (Riemann sums.) Fix $a < b$ in $\mathbb{R}$. We let

$$N = \left\{ (P, P^*) : \begin{array}{c} n \in \mathbb{N},\ P = \{a = t_0 < t_1 < \cdots < t_n = b\} \text{ (partitions)} \\ P^* = \{t_1^*, \ldots, t_n^*\} \text{ where each } t_j^* \in [t_{j-1}, t_j] \text{ (labels)} \end{array} \right\}$$

We use "refinement" ordering: $(P, P^*) \leq (Q, Q^*) \Leftrightarrow P \subseteq Q$. Given $f : [a, b] \to \mathbb{F}$ we define the *Riemman sum*: $f_{(P,P^*)} = \sum_{j=1}^{n} f(t_j^*)(t_j - t_{j-1}) \in \mathbb{F}$, where P, P^* are as in the description of N, above. Then $(f_{(P,P^*)})_{(P,P^*)\in N}$ is the Riemann sum net in $\mathbb{F}$.

(iv) (Nets from filtering families.) A famly $\mathcal{F} \subset \mathcal{P}(X) \setminus \{\varnothing\}$ is called *filtering* if for any F_1, F_2 in $\mathcal{F}$, there is F_3 in $\mathcal{F}$ so $F_3 \subseteq F_1 \cap F_2$. [We further call $\mathcal{F}$ is *filter* if for any F in $\mathcal{F}$ and $A \supseteq F$ we have $A \in \mathcal{F}$ as well. Hence ultrafilters are filters and thus filtering families.] We let

$$N_{\mathcal{F}} = \{(x, F) : x \in F,\ F \in \mathcal{F}\}, \text{ with preorder: } (x, F) \leq (x', F') \ \Leftrightarrow F \supseteq F'.$$

The symmetry and transitivity conditions are straightforward. The filtering condition implies that $N_{\mathcal{F}}$ is directed. Indeed, if $(x_1, F_1), (x_2, F_2) \in N_{\mathcal{F}}$ then for any F_3 in $\mathcal{F}$ with $F_3 \subseteq F_1 \cap F_2$ and any $x_3 \in F_3$, we have $(x_j, F_j) \leq (x_3, F_3)$ for $j = 1, 2$.

We then let $x_{(x,F)} = x$, so $(x)_{(x,F)\in N_{\mathcal{F}}}$ is the net created by $\mathcal{F}$. Notice that if $F \in \mathcal{F}$, then $(x)_{(x,F)\in N_{\mathcal{F}}}$ is eventually in F.

An *ultranet* is a net in X which, given any subset of X, is eventually in that subset, or in its complement. If $\mathcal{U}$ is an ultrafilter, then $(x)_{(x,U)\in N_{\mathcal{U}}}$ is an ultranet.

Nets and topology. Now we let (X, τ) be a topological space.

Definition. Given a net $(x_\nu)_{\nu\in N}$ and x_0 in X, we say that x_0 is a

- τ*-limit point* for $(x_\nu)_{\nu\in N}$ if for every $U \in \tau$ with $x_0 \in U$, we have that $(x_\nu)_{\nu\in N}$ is eventually in U; we write $x_0 = \tau\text{-}\lim_{\nu\in N} x_\nu$ $(*)$; and
- τ*-cluster point* for $(x_\nu)_{\nu\in N}$ if for every $U \in \tau$ with $x_0 \in U$, we have that $(x_\nu)_{\nu\in N}$ is frequently in U.

$(*)$ There is a slight technical problem with this notation, as the uniqueness of a limit is guaranteed only when τ is Hausdorff (exercise). Nevertheless, this notation is too convenient not to use.

Proposition. (Subnet characterization of cluster points.) *Given a net $(x_\nu)_{\nu\in N}$ and a point x_0 in X, we have that*

$$x_0 \text{ is a } \tau\text{-cluster point of } (x_\nu)_{\nu\in N} \Leftrightarrow \text{there is a subnet } (x_{\nu_\mu})_{\mu\in M} \text{ for which } x_0 = \tau\text{-}\lim_{\mu\in M} x_{\nu_\mu}.$$

Proof. ($\Rightarrow$) For each ν in N and $U\in\tau$ with $x_0\in U$ we let

$$F_{\nu,U} = \{\nu'\in N : \nu'\geq\nu,\ x_{\nu'}\in U\}$$

which is non-empty as x_0 is a cluster point. Then $\mathcal{F} = \{F_{\nu,U} : \nu\in N,\ U\in\tau \text{ with } x_0\in U\}$ is a filtering family. Indeed, we have for given $F_{(\nu_1,U_1)}, F_{(\nu_2,U_2)}$ in $\mathcal{F}$, that for any $\nu_3\geq\nu_1,\nu_2$ we have that $F_{\nu_3,U_1\cap U_2}\subseteq F_{(\nu_1,U_1)}\cap F_{(\nu_2,U_2)}$.

Then we let $N_{\mathcal{F}} = \{(\nu,F) : \nu\in F,\ F\in\mathcal{F}\}$ with pre-order as in (iv), above, and let $\nu_{(\nu,F)} = \nu$.

The map $(\nu,F)\mapsto\nu : N_{\mathcal{F}}\to N$ is cofinal: given ν_0 in N any $F_0 = F_{\nu_0,U}$ in $\mathcal{F}$ is comprised of points ν satisfying $\nu\geq\nu_0$, and hence any F in $\mathcal{F}$ with $F\subseteq F_0$ is comprised of such points too. Hence $(x_\nu)_{(\nu,F)\in N_{\mathcal{F}}}$ is a subnet of $(x_\nu)_{\nu\in N}$. Furthermore, if $U\in\tau$ with $x_0\in U$, then for any ν in N, and any $\nu'\in F$ where $F\subseteq F_{\nu,U}$, we have $x_{\nu'}\in U$, i.e. $(x_\nu)_{(\nu,F)\in N_{\mathcal{F}}}$ is eventually in U.

Proof of ($\Rightarrow$) using directed maps for subnets. We let $\tau_{x_0} = \{U\in\tau : x_0\in U\}$ which is directed by reverse inclusion: if $U,U'\in\tau_{x_0}$ then $U,U'\supseteq U\cap U'$ where $U\cap U'\in\tau_{x_0}$. Let

$$M = \{(\nu,U) : x_\nu\in U,\ U\in\tau \text{ with } x_0\in U\}$$

with preorder: $(\nu,U)\leq(\nu',U')\Leftrightarrow\nu\leq\nu'$ and $U\supseteq U'$. This is directed: if $(\nu_1,U_1),(\nu_2,U_2)\in M$, then there is $\nu_3'\geq\nu_1,\nu_2$ and then $\nu_3\geq\nu_3'$ such that $x_{\nu_3}\in U_1\cap U_2$, so $(\nu_3,U_1\cap U_2)\geq(\nu_j,U_j)$ for $j=1,2$ in M.

Let $\nu_{(\nu,U)}=\nu$. Notice that $(\nu,U)\mapsto\nu : M\to N$ is pre-order preserving, (dm1) above; and for any ν in N and $U\in\tau_{x_0}$ there is $\nu'\geq\nu$ in N so $\nu_{(\nu',U)}=\nu'\geq\nu$, (dm2) above. Hence $(\nu,U)\mapsto\nu : M\to N$ is a directed map, hence a cofinal map. Thus $(x_\nu)_{(\nu,U)\in M}$ is a subnet of $(x_\nu)_{\nu\in N}$. Given U in τ_{x_0}, $(x_\nu)_{(\nu,U)\in M}$ is specifically engineered to eventually be inside U. Indeed, for any ν for which $(\nu,U)\in M$, and any $(\nu',U')\geq(\nu,U)$ in M we have $x_{\nu'}\in U'\subseteq U$.

($\Leftarrow$) If for any U in τ with $x_0\in U$, a subnet $(x_{\nu_\mu})_{\mu\in M}$ is eventually in U, then $(x_\nu)_{\nu\in N}$ is frequently in U. □

Remark. It is worth reflecting on the subsequence characterization of cluster points of sequences in metric spaces. It relies on three points, any or all of which may fail in non-metrizable settings:

- $(\mathbb{N}, \leq)$ is well-ordered;
- any infinite subset of $\mathbb{N}$ is cofinal; and
- any point in a metric space has a sequential neighbourhood base.

Indeed, in a metric space (X, d) suppose x_0 is a cluster point of $(x_n)_{n=1}^{\infty}$. We let

- $n_1 = \min\{n : d(x_n, x_0) < 1\}$; then inductively
- $n_{k+1} = \min\{n : n > n_k \text{ and } d(x_n, x_0) < \frac{1}{k}\}$.

Hence $n_1 < n_2 < \dots$ and $d(x_{n_k}, x_0) < \frac{1}{k}$.

Proposition. (Subnet characterization of continuity.) *If (Y, σ) is another topological space, and $f : X \to Y$ then*

$$f \text{ is } \tau\text{-}\sigma\text{-continuous} \Leftrightarrow$$
$$f(x_0) = \sigma\text{-}\lim_{\nu \in N} f(x_\nu) \text{ in } Y \text{ whenever } x_0 = \tau\text{-}\lim_{\nu \in N} x_\nu \text{ in } X.$$

Proof. $(\Rightarrow)$ Given V in σ with $f(x_0)$ in V, we have that $U = f^{-1}(V)$ is open (A1,Q1) with x_0 in U. If $(x_\nu)_{\nu \in N}$ is eventually in U, then $(f(x_\nu))_{\nu \in N}$ is eventually in V.

$(\Leftarrow)$ Fix x_0 in X. We note that $\tau_{x_0} = \{U \in \tau : x_0 \in U\}$ is a filtering family as it is closed under pairwise intersection and no member is empty. Thus, as in (iv) in the examples above, we set $N_{\tau_{x_0}} = \{(x, U) : x \in U,\ U \in \tau_{x_0}\}$, which is a directed set with preorder: $(x, U) \leq (x', U') \Leftrightarrow U \supseteq U'$. We let $x_{(x,U)} = x$, so $(x)_{(x,U) \in N_{\tau_{x_0}}}$ is a net in X, specifically arranged so $x_0 = \tau\text{-}\lim_{(x,U) \in N_{\tau_{x_0}}} x$.

Hence our assumptions provide that $f(x_0) = \sigma\text{-}\lim_{(x,U) \in N_{\tau_{x_0}}} f(x)$. Hence given V in σ so $f(x_0) \in V$, i.e. V in $\sigma_{f(x_0)}$, there is ν_V in $N_{\tau_{x_0}}$ so $f(x_\nu) \in V$ for $\nu \geq \nu_V$. We write $\nu_V = (x, U)$. Then if $x' \in U$ we have $(x', U) \geq (x, U)$ so $f(x') = f(x_{(x',U)}) \in V$. But then $f(U) = \bigcup_{x' \in U} \{f(x')\} \subseteq V$, so f is continuous at x_0. But this may be done for arbitrary x_0 in X.

Proof of $(\Leftarrow)$ **by contrapositive.** We suppose that f is not continuous. Hence there is a V in σ for which $f^{-1}(V)$ is not open (A1,Q1). Thus there is $x_0 \in f^{-1}(V)$ for which no U in τ with x_0 in U satisfies $U \subseteq f^{-1}(V)$, in other words $\mathcal{F} = \{U \setminus f^{-1}(V) : U \in \tau \text{ with } x_0 \in U\}$ is a filtering family. As above, the net $(x)_{(x,F)\in N_{\mathcal{F}}}$ is designed to converge to x_0. However $\{f(x)\}_{(x,F)\in N_{\mathcal{F}}} \cap V = \varnothing$, so the net $(f(x))_{(x,F)\in N_{\mathcal{F}}}$ does not admit $f(x_0)$ as a limit point. $\square$

Consequences. (i) If τ', τ are both topologies on X, then

$$\tau' \subseteq \tau \;\Leftrightarrow\; x_0 = \tau'\text{-}\lim_{\nu\in N} x_\nu \text{ whenever } x_0 = \tau\text{-}\lim_{\nu\in N} x_\nu, \text{ in } X.$$

Indeed, $\tau' \subseteq \tau \Leftrightarrow \mathrm{id} : X \to X$ is τ-τ'-continuous (A1,Q1).

(ii) (Convergence in product spaces.) We consider the usual product topology: $X = \prod_{\alpha\in A} X_\alpha$ where $\{(X_\alpha, \tau_\alpha)\}_{\alpha\in A}$ is a collection of topological spaces, and let $\pi = \sigma(X, \{p_\alpha\}_{\alpha\in A})$ where each $p_\alpha : X \to X_\alpha$ is the coordinate projection. Then

$$x^{(0)} = \pi\text{-}\lim_{\nu\in N} x^{(\nu)} \;\Leftrightarrow\; x^{(0)}_\alpha = \tau_\alpha\text{-}\lim_{\nu\in N} x^{(\nu)}_\alpha \text{ for each } \alpha \text{ in } A.$$

Indeed, π is the coarsest topology making each $p_\alpha : X \to X_\alpha$, π-τ_α-continuous.

(ii') (Pointwise convergence in function spaces.) Let (X, τ) be a topological space, A a set, an consider the space X^A with product topology. We identify X^A with the set of functions mapping $A \to X$, by $(x_a)_{a\in A} \leftrightarrow f$, where $f : A \to X$ is given by $f(a) = x_a$. Then (ii) show that π is the topology of "pointwise convergence":

$$f_0 = \pi\text{-}\lim_{\nu\in N} f_\nu \;\Leftrightarrow\; f_0(a) = \tau\text{-}\lim_{\nu\in N} f_\nu(a) \text{ in } X \text{ for each } a \text{ in } A.$$

(ii") Let $\mathcal{X}$ be a normed space. Then on $\mathcal{X}^*$, $w^* = \sigma(\mathcal{X}^*, \widehat{\mathcal{X}})$ is the topology of pointwise convergence:

$$f_0 = w^*\text{-}\lim_{\nu\in N} f_\nu \;\Leftrightarrow\; f_0(x) = \lim_{\nu\in N} f_\nu(x) \text{ in } \mathbb{F} \text{ for each } a \text{ in } A.$$

Indeed, in proof of Alaoglu's Theorem we saw that the embedding $\mathcal{X}^* \hookrightarrow \mathbb{F}^{\mathcal{X}}$ is w^*-$\pi|_{\mathcal{X}^*}$-open onto its range. As (i), above, shows, this is a homeomorphism.