PMATH 453/753 , FALL 2019

Assignment #5 Due: December 2

1. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces.

   (a) $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is called *adjointable* if $T^* f \in \mathcal{X}^*$ for each $f \in \mathcal{Y}^*$. Prove that T is adjointable only if it is bounded.

   [Hint: Banach-Steinhaus.]

   (b) Show that F in $\mathcal{X}^{**}$ is w^*-continuous, i.e. as a function from $\mathcal{X}^*$ to $\mathbb{F}$, if and only if $F = \hat{x}$ for some x in $\mathcal{X}$.

   [Hint: look in proof of w^*-Separation Theorem.]

   (c) Show that $S \in \mathcal{B}(\mathcal{Y}^*, \mathcal{X}^*)$ is w^*-w^* continuous if and only if $S = T^*$ for some T in $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

2. Let $\mathcal{X}$ be a Banach space and $T, S \in \mathcal{B}(\mathcal{X})$.

   (a) Show that the series $\exp T := \sum_{n=0}^{\infty} \frac{1}{n!} T^n$ converges in norm.

   (b) Show that if $TS = ST$, then $\exp S \exp T = \exp(S + T)$.

   (c) Show that if $\mathcal{H}$ is a complex Hilbert space, and H in $\mathcal{B}(\mathcal{H})$ is Hermitian, i.e. $H^* = H$, then $U = \exp(iH)$ is a unitary operator, i.e. $UU^* = I = U^* U$.

3. Let $\mathcal{H}$ be a complex Hilbert space and $R, S, T \in \mathcal{B}(\mathcal{H})$ be so $R^* R = RR^*$, $T^* T = TT^*$ and $RS = ST$. Prove that $R^* S = ST^*$.

   [Hint: expand $\exp(R^* - R) S \exp(T - T^*)$ to show that $\|\exp(R^*) S \exp(-T^*)\| \leq S$; then apply Liouville's Theorem to the entire function $F_f(z) = f\big(\exp(zR^*) S \exp(-zT^*)\big)$ for any $f \in \mathcal{B}(\mathcal{H})^*$ to deduce that $\exp(zR^*) S = S \exp(zT^*)$.]

4. Let $\mathcal{H}$ be a Hilbert space and P in $\mathcal{B}(\mathcal{H})$ be an idempotent, i.e. $P^2 = P$. Show that the following are equivalent:

   (i) P is an orthogonal projection

   (ii) P is self-adjoint: $P^* = P$

   (iii) P is normal: $P^* P = PP^*$

   (iv) $(Px, x) = \|Px\|^2$ for every x in $\mathcal{H}$.

5. **(a)** If $\mathcal{X}$ is a complex Banach space and E in $\mathcal{B}(\mathcal{X})$ is an idempotent, $E^2 = E$ compute the spectrum $\sigma(E)$.

   (b) Let $a = (a_n)_{n=1}^{\infty}$ in ℓ_∞. Let the *multiplication operator* $M_a : \ell_p \to \ell_p$ $(1 \le p < \infty)$ be given by $M_a(x_n)_{n=1}^{\infty} = (a_n x_n)_{n=1}^{\infty}$. Compute

$$\|M_a\|, \quad \sigma_p(M_a) \quad \text{and} \quad \sigma(M_a).$$

   (c) If K is a compact subset of $\mathbb{C}$, show that there is an operator $T \in \mathcal{B}(\ell_p)$ for which $\sigma(T) = K$.

   (d) Show that M_a is compact on ℓ_p if and only if $a \in \boldsymbol{c}_0$.

6. Let $q(t) = t^2 - 1$, $p_n = D^n[q^n]$ for $n = 0, 1, 2 \ldots$, where D is the differentiation operator and let $\tilde{p}_n = \frac{1}{\|p_n\|_2} p_n$, where $\|\cdot\|_2$ is the norm in the Hilbert space $L_2[-1,1]$.

   (a) Let

$$\mathcal{D} = \left\{ f \in L_2[-1,1] : \sum_{k=1}^{\infty} |k(k+1)(f, \tilde{p}_k)|^2 < \infty \right\}.$$

   Show that the operator $L_0(p) = D[q\, Dp]$ on polynomials extends to an operator $L : \mathcal{D} \to L_2[-1,1]$, with closed graph in $L_2[-1,1] \oplus_2 L_2[-1,1]$.

   [Hint. First, find a new inner product on $\mathcal{D}$ which makes $\mathcal{D}$ a Hilbert space in which $\text{span}\{p_n\}_{n=1}^{\infty}$ is dense, and allows L to be viewed as a bounded operator.]

   (b) Given $\varepsilon > 0$, show that there is a compact linear Hermitian operator K_ε on $L_2[-1,1]$ for which $\text{ran} K = \mathcal{D}$ and

$$K_\varepsilon(L + \varepsilon I) f = f \text{ for } f \text{ in } \mathcal{D}, \text{ while } (L + \varepsilon I) K_\varepsilon = I \text{ on } L_2[-1,1].$$

   [Hint: Notice that $(L + \varepsilon I) p_n = (n^2 + n + \varepsilon) p_n$. Consider Q5(d), above.]