PMATH 453/753 , FALL 2019

Assignment #4 Due: November 15

1. Compute the set of *extreme points* of the unit ball, $\mathrm{ext}B(\mathcal{X})$, for the following $\mathbb{R}$-Banach spaces $\mathcal{X}$, i.e. don't worry about the complex cases.

   (a) $\mathcal{X} = L_1[0,1]$ [Hint: show that $F(s) = \int_{[0,s]} |f|$ is continuous.]

   (b) $\mathcal{X} = \ell_1$

   (c) $\mathcal{X} = \ell_\infty$

   (d) Show that $\mathrm{co}\,\mathrm{ext}B(\ell_1)$ is $\|\cdot\|_1$-dense in $B(\ell_1)$.

2. Let $\mathcal{C} = \mathcal{C}^{\mathbb{R}}[0,1]$ and $P[0,1] = \{\mu \in B(\mathcal{C}^*) : \mu(\mathbf{1}) = 1\}$.

   (a) Show that

$$\overline{\mathrm{co}}\,\mathrm{ext}P[0,1] = \left\{ \sum_{j=1}^{\infty} t_j \widehat{x_j} : \{x_j\}_{j=1}^{\infty} \subset [0,1],\text{ each } t_j \geq 0 \text{ and } \sum_{j=1}^{\infty} t_j = 1 \right\}.$$

   (b) Let $\mu : \mathcal{C} \to \mathbb{R}$ be given by $\mu(f) = \int_0^1 f$ (Riemann integral). Show that $\mu \in P[0,1] \setminus \overline{\mathrm{co}}\,\mathrm{ext}P[0,1]$. Deduce that $\overline{\mathrm{co}}\,\mathrm{ext}P[0,1] \subsetneq \overline{\mathrm{co}}^{w*}\,\mathrm{ext}P[0,1]$.

   [Hint: show for ν in $\overline{\mathrm{co}}\,\mathrm{ext}P[0,1]$, that there is a sequence $(f_n)_{n=1}^{\infty} \subset S(\mathcal{C})$ such that $\mu(f_n) \to 0$ while $\nu(f_n)$ does not.]

3. Let $B_{\mathbb{R}} = B\left(\mathcal{C}^{\mathbb{R}}[0,1]\right)$ and $B_{\mathbb{C}} = B\left(\mathcal{C}^{\mathbb{C}}[0,1]\right)$.

   (a) Show that $\mathrm{ext}B_{\mathbb{R}} = \{-\mathbf{1}, \mathbf{1}\}$ where $\mathbf{1}$ is the constant function with range 1.

   (b) Deduce that $\mathcal{C}^{\mathbb{R}}[0,1]$ is not isometrically isomorphic to a dual space.

   (c) Show that $\mathrm{ext}B_{\mathbb{C}} = \mathcal{C}^{\mathbb{S}}[0,1] := \{f \in \mathcal{C}^{\mathbb{C}}[0,1] : |f(t)| = 1 \text{ for each } t \text{ in } [0,1]\}$.

   (d) Show that $\mathrm{co}\,\mathcal{C}^{\mathbb{S}}[0,1]$ is $\|\cdot\|_\infty$-dense in $B_{\mathbb{C}}$.

   [Hint: if $z \in \mathbb{C},\ 0 < |z| \leq 1$ then $z = \frac{1}{2}\mathrm{sgn}z(|z| + i\sqrt{1-|z|^2} + |z| - i\sqrt{1-|z|^2})$ – draw a picture with $|z|$, first, to see how one gets this. Thus if $f \in B_{\mathbb{C}}$, $|f(t)| > 0$ for all t, we can show that $f \in \mathrm{co}\,\mathcal{C}^{\mathbb{S}}[0,1]$.]

   (e) (Bonus) Does $\mathcal{C}^{\mathbb{C}}[0,1]$ admit an isometric predual?

4. A sequence $(x_n)_{n=1}^\infty$ in a Banach space $\mathcal{X}$ is called *null* if $\lim_{n\to\infty} x_n = 0$ (norm limit).

   (a) Show that if $(x_n)_{n=1}^\infty$ is a null sequence, then its norm-closed convex hull $C = \overline{\text{co}}\{x_n\}_{n=1}^\infty$ is norm-compact.

   [From PMath 351, it suffices to show that C is *totally bounded*.]

   (b) Show that a closed set K in a Banach space $\mathcal{X}$ is norm-compact, if and only if there is a null sequence $\{x_n\}_{n=1}^\infty$ such that $K \subset \overline{\text{co}}\{x_n\}_{n=1}^\infty$. Deduce that the norm-closed convex hull of a compact set in a Banach space is compact.

   [Note that $2K$ is compact; cover with closed 1/2-balls, $y_i + \frac{1}{2}B(\mathcal{X})$. Do this again with $K_1 = \bigcup_{i=1}^n([2K \cap (y_i + \frac{1}{2}B(\mathcal{X}))] - y_i)$. Continue.]

5. **(a)** Let $\mathcal{H}$ be a Hilbert space and $P \in \mathcal{B}(\mathcal{H})$ satisfy $P^2 = P$ and $\|P\| = 1$. Show that P is the orthogonal projection onto $\mathcal{M} = \text{ran}P$.

   [Hint. First establish that $[\ker P]^\perp = [\text{ran}(I - P)]^\perp \subseteq \mathcal{M}$. Then use uniqueness of orthogonal decomposition]

   (b) Exhibit a Hilbert space $\mathcal{H}$ and a projection $P = P^2$ in $\mathcal{B}(\mathcal{H})$ for which $\|P\| > 1$.

6. Let $q(t) = t^2 - 1$ and define the family of (non-normalised) *Legendre polynomials* by $p_n = D^n[q^n]$ where D is the differentiation operator.

   (a) Show that $(p_n)_{n=0}^\infty$ is an orthogonal sequence in $L_2[-1, 1]$. [Hint: integration by parts, $D^k[q^n](\pm 1) = 0$ if $0 \leq k < n$.]

   (b) Deduce that $(\tilde{p}_n)_{n=0}^\infty$, where each $\tilde{p}_n = \frac{1}{\|p_n\|_2} p_n$, is the ortho-normal sequence found by applying Gram-Schmdit orthogonalisation to the sequence of basic monomials $(m_n)_{n=0}^\infty$, $m_n(t) = t^n$; hence $(p_n)_{n=0}^\infty$ is a *complete* orthogonal set, i.e. its linear span is dense.

   (c) Show that p_n satisfies $(t^2 - 1)p''(t) + 2tp'(t) = n(n+1)p(t)$, the nth *Legendre equation.* [Hint: $D[q\, Dp_n] \perp p_k$ for $k < n$.]

   (d) Let $\mathcal{D} = \{f \in L_2[-1, 1] : \sum_{k=1}^\infty |k(k+1)(f, \tilde{p}_k)|^2 < \infty\}$. Show that the operator $L_0(p) = [q\, D^2 + 2m_1\, D]p$ on polynomials extends to an operator $L : \mathcal{D} \to L_2[-1, 1]$, with closed graph in $L_2[-1, 1] \oplus_2 L_2[-1, 1]$.

   [Hint. First, find a new inner product on $\mathcal{D}$ which makes $\mathcal{D}$ a Hilbert space in which $\text{span}\{p_n\}_{n=1}^\infty$ is dense, and allows L to be viewed as a bounded operator.]