PMATH 453/753 , FALL 2019

Assignment #3 Due: October 28

Given a normed space $\mathcal{X}$ with dual $\mathcal{X}^*$ we always let the weak topology on $\mathcal{X}$ be given by $w = \sigma(\mathcal{X}, \mathcal{X}^*)$, and the weak* topology on $\mathcal{X}^*$ be given by $w^* = \sigma(\mathcal{X}^*, \widehat{\mathcal{X}})$. Recall that $B(\mathcal{X}) = \{x \in \mathcal{X} : \|x\| \leq 1\}$ while $S(\mathcal{X}) = \{x \in \mathcal{X} : \|x\| = 1\}$.

1. Show that the followings sets are pairwise *homeomorphic*, i.e. there exists a continuous open bijection between any pair of them.

   (i) $\big(\{\chi_E : E \in \mathcal{P}(\mathbb{N})\}, w^*\big)$, in $\ell_\infty \cong {\ell_1}^*$.

   (ii) $\big(\{0,1\}^{\mathbb{N}}, \pi\big)$, π is the product topology.

   (iii) (C, τ), C is "middle thirds" Cantor set in $[0,1]$, τ the usual metric/order topology.

2. Show that if $(B(\mathcal{X}^*), w^*|_{B(\mathcal{X}^*)})$ is *metrisable* – i.e. there exists a metric on $B(\mathcal{X}^*)$ which gives realtivized weak* topology – then $\mathcal{X}$ must be separable.

   [Hint: If $w^*|_{B(\mathcal{X}^*)}$ is metrisable, then there must be a countable neighbourhood base at 0.]

3. Let $\mathcal{X}$ be a normed space. Show that:

   (a) If $\mathcal{X}$ is infinite-dimensional, then $\overline{S(\mathcal{X})}^w = B(\mathcal{X})$.

   [Hint: Begin by showing that every neighbourhood of 0 contains a subspace. Consider various characterisations of the closure.]

   (b) Deduce that $\mathcal{X}$ is finite-dimensional $\Leftrightarrow w = \tau_{\|\cdot\|}$.

4. Let $1 < p < \infty$.

   (a) Let $(x^{(n)})_{n=1}^\infty$ be a sequence in ℓ_p and $x^{(0)} \in \ell_p$. Show that

   $$w\text{-}\lim_{n\to\infty} x^{(n)} = x^{(0)} \quad \iff \quad \begin{array}{c} \lim_{n\to\infty} x_i^{(n)} = x_i^{(0)} \text{ for each } i \\ \text{and } \sup_{n\in\mathbb{N}} \left\|x^{(n)}\right\|_p < \infty. \end{array}$$

   (b) Must a weakly converging net in $(x^{(\nu)})_{\nu\in N}$ in ℓ_p be *cofinally bounded*: i.e. must there be a ν_0 for which $\sup_{\nu\geq\nu_0} \|x^{(\nu)}\|_p < \infty$?

   [Hint: you method for solving Q3 (a) may help.]

5. Let (X, τ) be a topological space. Show that the following are equivalent

   (i) (X, τ) is compact;

   (ii) every net in X admits a τ-cluster point; and

   (iii) every ultranet in X admits a τ-limit point.

The next 2 questions address the inadequacy of sequences for judging if sets are compact or closed, in non-metrizable setings.

6. A topological space (X, τ) is called *sequentially compact* if for every sequence in X admits a τ-converging subsequence. Compact metric spaces are sequentially compact from PMATH 351.

   (a) Let $I = [0, 1]$ and consider (I^I, π) (product topology). Show that (I^I, π) is not sequentially compact.

   [Hint: consider $x_t^{(n)} = 3^n t - \lfloor 3^n t \rfloor$.]

   (b) Why does this not contradict Q5 (a)?

   (c) (Bonus) Exhibit a cluster point of the sequence you used to prove part (a), above.

7. **(a)** Prove that if $(x^{(n)})_{n=1}^{\infty}$ is a sequence in ℓ_1, which converges weakly to $x^{(0)}$ in ℓ_1, then $\lim_{n\to\infty} \left\| x^{(n)} - x^{(0)} \right\|_1 = 0$.

   [Hint: If $\sigma(\ell_1, \widehat{\boldsymbol{c}_0})$-$\lim_{n\to\infty} x^{(n)} = 0$ but $\left\| x^{(n)} \right\|_1 \geq \varepsilon > 0$, then extract a subsequence $(x^{(n_k)})_{k=1}^{\infty}$ which does not converge weakly to 0.]

   (b) Exhibit a net in $S(\ell_1)$ which converges weakly to 0.

   [Hint: Q3 (a) tells us that such a net exists. The method of solution can help find one.]