PMATH 453/653 (AMATH 423), FALL 2019

Assignment #2 Due: October 4

1. Let $\mathcal{X}$ be a $\mathbb{R}$-normed space.

   (a) Let $\mathcal{Y}$ a closed subspaceof $\mathcal{X}$, and $x_0 \in \mathcal{X} \setminus \mathcal{Y}$. Show that there is $f \in \mathcal{X}^*$ such that $\mathcal{Y} \subseteq \ker f$ and $f(x_0) = \mathrm{dist}(x_0, \mathcal{Y})$.

   (b) Show that if $\mathcal{X}^*$ is *separable*, i.e., there is a countable dense subset in $\mathcal{X}^*$, then $\mathcal{X}$ must be separable too.

   (c) If $\mathcal{X}$ is separable, must $\mathcal{X}^*$ also be separable? Prove your assertion.

2. Let $\ell_\infty = \ell_\infty^{\mathbb{R}}$, with uniform norm $\|\cdot\|_\infty$. This exercise describes Banach's *generalised limits* of bounded sequences.

   (a) Show that if $\mathcal{Y}$ is any subspace of ℓ_∞, then the functional $p : \ell_\infty \to \mathbb{R}$, $p(x) = \mathrm{dist}(x, \mathcal{Y})$, is sublinear, with $p(x) \leq \|x\|_\infty$ for every x.

   (b) Show that there exists a linear functional $L : \ell_\infty \to \mathbb{R}$ such that

   (i) $\|L\| = 1$ and $L(\mathbf{1}) = 1$, where $\mathbf{1} = (1, 1, \dots)$, and

   (ii) $L(T_n x) = L(x)$, where $T_n x = (x_{n+1}, x_{n+2}, \dots)$ if $x \in \ell_\infty, n \in \mathbb{N}$.

   [Hint: consider $\mathcal{Y} = \mathrm{span}\{x - T_1 x : x \in \ell_\infty\}$.]

   (c) Show that $\liminf_{n\to\infty} x_n \leq L(x) \leq \limsup_{n\to\infty} x_n$.

   [Hint: first show $L|_{\mathbf{c}_0} = 0$; $L(x) \geq 0$ if $x_n \geq 0$.]

   (d) Fix $m \in \mathbb{N}$ and let $x = (n/m - \lfloor n/m \rfloor)_{n=1}^\infty$, where $\lfloor s \rfloor = \max\{k \in \mathbb{N} : k \leq s\}$ for s in $\mathbb{R}$. Compute $L(x)$. [Hint: this sequence is periodic.]

If $\mathcal{X}$ is a vector space over $\mathbb{F}$, a *Hamel basis* is any subset B which is:

• *linearly independant:* every finite subset of B is linearly independant;

• *spanning:* span$B$, the space of finite linear combinations of elements from B, is all of $\mathcal{X}$.

3. Show that an infinite dimensional Banach space cannot have a countable Hamel basis. [Hint: Baire.]

4. Let $1 \leq p < \infty$.

   (a) Show that $|\ell_p| = \mathfrak{c}$, i.e. the cardinality of ℓ_p is that of the contiuum.

   (b) Show that there exists a family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ with the following properties:
   (i) if $E, F \in \mathcal{F}$ with $E \neq F$, then $E \cap F$ is finite (or empty),
   (ii) $|\mathcal{F}| = \mathfrak{c}$.
   [Hint: the solution seems irrational.]

   (c) Show, without using the continuum hypothesis, that ℓ_p admits a Hamel basis of cardinality $\mathfrak{c}$.

   (d) Show that any Hamel basis for ℓ_p must have cardinality $\mathfrak{c}$.

   See the text, *Linear analysis*, p. 83, for a notion of countable bases on some Banach spaces. I will entertain submissions on q. 16 for bonus credit.

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5. Let $\mathcal{X}$ be an $\mathbb{F}$-vector space, and $\|\cdot\|$ and $|||\cdot|||$ each be norms under which $\mathcal{X}$ is complete.

   (a) Show that $\|\cdot\|$ and $|||\cdot|||$ are either *equivalent*, i.e. there are $m, M > 0$ such that

   $$m\|x\| \leq |||x||| \leq M\|x\| \quad \text{for each } x \text{ in } \mathcal{X}$$

   or their topologies satisfy

   $$\tau_{\|\cdot\|} \not\subset \tau_{|||\cdot|||} \text{ and } \tau_{|||\cdot|||} \not\subset \tau_{\|\cdot\|}. \qquad (\clubsuit)$$

   (b) Is it possible for ($\clubsuit$) to occur, with complete norms?

6. Let $1 < p, q < \infty$ (we do not suppose these to be conjugate). Suppose $A = [a_{ij}]_{i,j=1}^{\infty}$ is an infinite matrix with the property that

   $$Ax = \left(\sum_{j=1}^{\infty} a_{1j}x_j, \sum_{j=1}^{\infty} a_{2j}x_j, \dots\right) \in \ell_q$$

   for each $x = (x_1, x_2, \dots)$ in ℓ_p. Show that A, i.e. $x \mapsto Ax$, is a bounded operator from ℓ_p to ℓ_q.

   [Hint: you might want to warm-up by checking that the rows of A must be $\ell_{p'}$-sequences, $\frac{1}{p} + \frac{1}{p'} = 1$.]