PMATH 453/753, FALL 2019

Assignment #1 Due: September 20

1. Let (X, σ) and (Y, τ) be topological spaces and $f : X \to Y$ be a function. For any subset E of Y we let $f^{-1}(E) = \{x \in X : f(x) \in E\}$; thus $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ is a set function.

   As with metric topology, we define a subset E of X to be σ-closed, or simply *closed*, if $X \setminus E \in \sigma$, i.e., its compliment is open.

   Show that the following are equivalent:

   (i) f is σ-τ continuous at every point in X.

   (ii) $f^{-1}(V)$ is open for each open subset V of Y.

   (iii) $f^{-1}(F)$ is closed for each closed subset F of Y.

2. Let $p, q > 1$ be so $\frac{1}{p} + \frac{1}{q} = 1$; $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Exercises (a) and (b) show that there is a linear isometric isomorphism: ${\ell_p}^* \cong \ell_q$.

   (a) Show that if $b = (b_j)_{j=1}^{\infty} \in \ell_q$, then the map $f_b : \ell_p \to \mathbb{F}$ given by

   $$f_b\big((x_j)_{j=1}^{\infty}\big) = \sum_{j=1}^{\infty} x_j b_j$$

   defines a linear functional which is bounded with $\|f_b\| = \|b\|_q$.

   (b) Show that every bounded linear functional on ℓ_p arises as in (a), above.

   (c) Deduce that $(\ell_p, \|\cdot\|_p)$ is complete.

   (d) Determine which sequence space (if any) isometrically describes ${\ell_1}^*$. Prove your assertion.

3. Let (X, d) be a metric space. Recall that

$$\operatorname{Lip}(X, d) = \left\{ f : X \to \mathbb{F} \,\middle|\, f \text{ is bounded } \& \ L(f) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x, y)} < \infty \right\}$$

with norm $\|f\|_{\text{Lip}} = \|f\|_\infty + L(f)$.

(a) If $z \in X$, let $f_z : X \to \mathbb{F}$ be given by $f_z(x) = \min\{d(x, z), 1\}$. Show that $f_z \in \operatorname{Lip}(X, d)$ with $\|f_z\|_{\text{Lip}} \leq 2$.

(b) Let for x in X, $e_x : \operatorname{Lip}(X, d) \to \mathbb{F}$ be given by $e_x(f) = f(x)$. Verify that $e_x \in \operatorname{Lip}(X, d)^*$, and show that if (X, d) has bounded diameter, i.e. $\sup_{x,y \in X} d(x, y) < \infty$, then there is $c > 0$ such that

$$cd(x, y) \leq \|e_x - e_y\| \leq d(x, y) \text{ for } x, y \text{ in } X.$$

Hence, $d'(x, y) = \|e_x - e_y\|$ defines a metric on X which is equivalent to d.

(b') Show, without the assumption of bounded diameter, that the topology generated by the metric d' above is the same as that generated by d, i.e. $\tau_{d'} = \tau_d$.

In other words, the Banach space $(\operatorname{Lip}(X, d), \|\cdot\|_{\text{Lip}})$ remembers information about the metric on X.

On the other hand, $(\mathcal{C}_b(X), \|\cdot\|_\infty)$ does not do such a good job at remembering the metric.

(c) Now let for x in X, $e_x : \mathcal{C}_b(X) \to \mathbb{F}$ be given by $e_x(f) = f(x)$. Show, in this context, that

$$\|e_x - e_y\| = 2 \text{ if } x \neq y \text{ in } X.$$

Notice that $\rho(x, y) = \|e_x - e_y\|$ is always (equivalent to) the discrete metric in this setting.

Don't forget problem 4 ...

4. This exercise is concerned with computing the dual of space of the bounded sequences, $\ell_\infty = \{x \in \mathbb{F}^{\mathbb{N}} \mid \|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| < \infty\}$.

(a) If $E \subset \mathbb{N}$, let χ_E denote the sequence with $\chi_{E,i} = 1$ if $i \in E$, and $\chi_{E,i} = 0$ otherwise. Show that the space of simple sequences, $\mathcal{S} = \text{span}\{\chi_E : E \subseteq \mathbb{N}\}$, is dense in ℓ_∞.

[For any set $\mathcal{E}$ in a vector space $\mathcal{X}$, we let $\text{span}\mathcal{E}$ denote the smallest subspace which contains $\mathcal{E}$.]

(b) Let $\mathcal{FA}(\mathbb{N})$ denote the space of all functions $\mu : \mathcal{P}(\mathbb{N}) \to \mathbb{F}$ which satisfy

(i) *finite additivity*: $\mu(E \cup F) = \mu(E) + \mu(F)$ if $E \cap F = \varnothing$, and

(ii) *bounded variation*:

$$\|\mu\|_{\text{var}} = \sup\left\{ \sum_{j=1}^{n} |\mu(E_j)| : \begin{array}{c} E_1, \ldots, E_n \subseteq \mathbb{N} \text{ and} \\ E_j \cap E_k = \varnothing \text{ if } j \neq k \end{array} \right\} < \infty.$$

Show that each μ in $\mathcal{FA}(\mathbb{N})$ determines a well-defined linear functional $f_\mu : \mathcal{S} \to \mathbb{F}$ by

$$f_\mu(x) = \sum_{j=1}^{n} \alpha_j \mu(E_j) \quad \text{whenever} \quad \begin{array}{c} x = \sum_{j=1}^{n} \alpha_j \chi_{E_j} \text{ and} \\ E_j \cap E_k = \varnothing \Leftrightarrow j \neq k. \end{array}$$

Moreover, this functional is bounded on the normed vector space $(\mathcal{S}, \|\cdot\|_\infty)$ with $\|f_\mu\| = \|\mu\|_{\text{var}}$.

Thus, by uniform continuity, the functional f_μ extends uniquely to a bounded linear functional on ℓ_∞. [We hence readily deduce that $\mathcal{FA}(\mathbb{N})$ is a normed vector space with pointwise operations: $(\mu + \alpha\nu)(E) = \mu(E) + \alpha\nu(E)$, for μ, ν in $\mathcal{FA}(\mathbb{N})$, α in $\mathbb{F}$ and $E \subseteq \mathbb{N}$; and the norm $\|\cdot\|_{\text{var}}$.]

(c) Show that every bounded linear functional on ℓ_∞ arises as above. Hence there is a linear isometric isomorphism: ${\ell_\infty}^* \cong \mathcal{FA}(\mathbb{N})$.

(d) (Bonus question) Is every element of $\mathcal{FA}(\mathbb{N})$ of the form $\mu(E) = \sum_{j\in E} \mu_j$ with $\sum_{j=1}^{\infty} |\mu_j| < \infty$? You must prove your assertion.