PMATH 451/651, Winter 2019

Assignment #6, Not to be handed in.

1. Define, for a Borel set $E \subset \mathbb{R}^d$ and x in $\mathbb{R}^d$, the *upper* and *lower densities*

$$\overline{D}_E(x) = \limsup_{r\to 0^+} \frac{\lambda(E \cap B_r(x)}{\lambda(B_r(x))}, \ \underline{D}_E(x) = \liminf_{r\to 0^+} \frac{\lambda(E \cap Br(x)}{\lambda(B_r(x))}$$

where $\lambda = \lambda_d$ is the Lebesgue measure.

(a) Show that for λ-a.e. x in E we have $\overline{D}_E(x) = \underline{D}_E(x) = 1$, while for λ-a.e. x in $\mathbb{R}^d \setminus E$ we have $\overline{D}_E(x) = \underline{D}_E(x) = 0$.

(b) Construct, for any pair $0 < \alpha \leq \beta < 1$, a Borel set $E \subset \mathbb{R}^d$ and a point x for which $\underline{D}_E(x) = \alpha$ while $\overline{D}_E(x) = \beta$. [You may choose $d = 1$ or $d = 2$, if you wish.]

(Bonus) Obtain (b) as above, but with choices $0 = \alpha < \beta = 1$, $0 = \alpha < \beta < 1$ and $0 < \alpha < \beta = 1$.

2. Consider the following spaces of functions on $\mathbb{R}$:

$Lip(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \,|\, |f(y) - f(x)| \leq M|y - x| \text{ for } x, y \in \mathbb{R}\}$ (Lipschitz)
$AC(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \,|\, f \text{ is absolutely continuous}\}$
$UC(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \,|\, f \text{ is uniformly continuous}\}$
$BV(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \,|\, f \text{ is of bounded variation}\}$

Establish for each function below, which of the above spaces each function is an element.

(a) $f(x) = 1_{[0,1]}(x)\sqrt{x} + 1_{(1,\infty)}$

(b) $g(x) = 1_{(0,1/\pi]}(x)x \sin\left(\frac{1}{x}\right)$

(c) $h(x) = 1_{(0,1/\pi]}(x)x^2 \sin\left(\frac{1}{x}\right)$

(d) Cantor ternary function φ.

3. **(a)** Let $(X, \mathcal{M}, \nu)$ be a complex measure space for which $\big\{\{x\} : x \in X\big\} \subset \mathcal{M}$. Show that the set of *discrete points* $D_\nu = \{x \in X : \nu(\{x\}) \neq 0\}$ satisfies $\sum_{x \in D_\nu} |\nu(\{x\})| < \infty$. Hence show that $\nu_d = \sum_{x \in D_\nu} \nu(\{x\})\delta_x$ defines a complex measure, and that $\nu_d \perp \nu_c$, where $\nu_c = \nu - \nu_d$.

   We call ν_d the *discrete part* and ν_c the *continuous part* of ν.

   (b) Let $F \in BV_r(\mathbb{R})$. Show that the following decomposition holds: $F = F_d + F_{cs} + F_{ac}$, where

   • $F_d = \sum_{a \in D_F} [F(a) - F(a^-)] H_a$ is a uniformly converging series of Heaviside functions ($H_a = \chi_{[a,\infty)}$) over the set D_F of discontinuities of F;

   • F_{sc} is continuous, with $F'_{sc} = 0$ λ-a.e. (the *singular-continuous* part); and

   • F_{ac} is absolutely continuous.

   Moreover, each $F_d, F_{sc}, F_{ac} \in BV_r$ with $T_F = T_{F_d} + T_{F_{sc}} + T_{F_{ac}}$.