PMATH 451/651, Winter 2019

Assignment #5, Due Fri., Apr. 5.

We shall always let $(X, \mathcal{M}, \mu)$ denote a measure space.

1. (Containment relations of L^p-spaces.) Suppose $1 \leq p < q < \infty$.

   (a) Show that $L^p(\mu) \not\subset L^q(\mu) \Leftrightarrow (X, \mathcal{M}, \mu)$ admits sets of arbitrary small positive measure.

   (b) Show that $L^q(\mu) \not\subset L^p(\mu) \Leftrightarrow (X, \mathcal{M}, \mu)$ admits sets of arbitrary large finite measure.

2. (Geometry of L^p spaces.) Let $(L, \|\cdot\|)$ be a normed space. A point f in L with $\|f\| = 1$ is called an *extreme point* of the unit ball if

   $$f = (1-t)f_1 + tf_2,\ 0 < t < 1 \text{ and } \|f_1\|, \|f_2\| \leq 1 \quad \Rightarrow \quad f_1 = f_2$$

   i.e. f cannot be represented as a proper convex combination of other elements of the unit ball.

   (a) Let $1 < p < \infty$. Show that every norm 1 element of the unit ball of $L^p(\mu)$ is an extreme point.

   (b) An *atom* for μ is a set A in $\mathcal{M}$ such that $\mu(A) > 0$ and any subset $E \subseteq A$, $E \in \mathcal{M}$ satisfies $\mu(E) = \mu(A)$ or $\mu(E) = 0$.

   Show that the unit ball of $L^1(\mu)$ admits an extreme point $\Leftrightarrow \mu$ admits an atom of finite measure.

3. **(a)** (Riemann-Stieltjes integration) Let $F : \mathbb{R} \to \mathbb{R}$ be non-decreasing and right continuous, and $f \in \mathcal{C}_c(\mathbb{R})$ Show that the sequence of F-*Riemann sums*

   $$S_n(f) = \sum_{k=-n2^n+1}^{n2^n} f(\tfrac{k}{2^n})[F(\tfrac{k}{2^n}) - F(\tfrac{k-1}{2^n})]$$

   Is Cauchy, and hence converges to a quantity $I_F(f)$. Furthermore, $I_F : \mathcal{C}_c(\mathbb{R}) \to \mathbb{C}$ defines a positive linear functional.

   (b) Determine the measure corresponding to I_F, as provided by Riesz representation theorem.

(c) (d-dimensional Riemann integration) Let $f \in \mathcal{C}_c(\mathbb{R}^d)$. Show that the sequence of *Riemann sums*

$$S_n(f) = \frac{1}{2^{dn}} \sum_{k_1=-n2^n+1}^{n2^n} \cdots \sum_{k_d=-n2^n+1}^{n2^n} f(\tfrac{k_1}{2^n}, \ldots, \tfrac{k_d}{2^n})$$

converges to $\int_{\mathbb{R}^d} f \, d\lambda_d$ (integral with respect to d-dimensional Lebesgue mesure).

(d) Let $\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{C}$ be a complex measure. Show that

$$\mu_n = \sum_{k=-n2^n+1}^{n2^n} \mu\left(\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]\right) \delta_{k/2^n}$$

converges *weak** to μ in the sense that

$$\lim_{n\to\infty} \int_X f \, d\mu_n = \int_X f \, d\mu \text{ for } f \text{ in } C_0(\mathbb{R}).$$

4. Let S be an uncountable set. Consider the metric d on $X = [0,1] \times S$ given by

$$d((x,s),(y,t)) = \begin{cases} |x-y| & \text{if } s = t \\ 1 & \text{if } s \neq t. \end{cases}$$

Observe that each set $[0,1] \times \{s\}$ is open. Let $I : C_c(X) \to \mathbb{C}$ be given by

$$I(f) = \sum_{s \in S} \int_0^1 f(x,s) \, dx \quad \text{(sum of Riemann integrals)}$$

where we observe the above sum is finite since $\text{supp}(f)$ is compact. Let $\mu : \mathcal{B}(X) \to [0,\infty]$ be the Radon measure for which $I(f) = \int_X f \, d\mu$.

(a) Let $E \in \mathcal{B}(X)$. Show that E is σ-finite with respect to μ if and only if $E \subseteq \bigcup_{k=1}^\infty([0,1] \times \{s_k\})$ for some $\{s_k\}_{k=1}^\infty \subset S$ (i.e. E meets only countably many components of X). Moreover, such a set $E \in \mathcal{B}([0,1]) \otimes \mathcal{P}(S)$ and we have $\mu(E) = \lambda \times \gamma(E)$ (product measure of Lebesgue and counting measures).

(b) Let $E = \{0\} \times S$. Show that E is Borel. Also show that E is not inner regular for μ. Moreover, show that $\lambda \times \gamma$ is locally finite on $\mathcal{B}([0,1]) \otimes \mathcal{P}(S)$ but not outer regular.

(Bonus) Determine whether or not $\mathcal{B}(X) = \mathcal{B}([0,1]) \otimes \mathcal{P}(S)$.