PMATH 451/651, Winter 2019

Assignment #4 Due: Fri., Mar. 15.

For the purposes of this assignment let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ denote the Lebesgue measure space.

1. (An infinite product measure.) Consider the infinite product set, $X = \{0,1\}^{\mathbb{N}} = \{0,1\} \times \{0,1\} \times \dots$.

   (a) Show that the metric space (X, d), with d given by $d(x,y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$ (where $x = (x_i)_{i=1}^{\infty}, y = (y_i)_{i=1}^{\infty} \in X$), is compact.

   (b) Show that the algebra $\mathcal{A}$ on X generated by the family of elementary sets

   $$\mathcal{E} = \big\{E_1 \times \dots \times E_n \times \{0,1\} \times \{0,1\} \times \dots : n \in \mathbb{N}, E_1, \dots, E_n \subseteq \{0,1\}\big\}$$

   is exactly the family of simultaneously closed and open ("clopen") subsets of X. Deduce that each element of $\mathcal{A}$ is a pairwise disjoint union

   $$A = \dot{\bigcup_{j=1}^{m}} (E_{j1} \times \dots \times E_{jn} \times \{0,1\} \times \{0,1\} \times \dots) \qquad (\dagger)$$

   where $m, n \in \mathbb{N}$ and each $E_{ji} \subseteq \{0,1\}$. [Hint: first study open balls $B_r(x)$, $r > 0$; compactness is our friend.]

   (c) Show that $\mu_0 : \mathcal{A} \to [0,1]$, given on set A, as in $(\dagger)$, by

   $$\mu_0(A) = \sum_{j=1}^{m} \frac{1}{2^n} |E_{j1}| \dots |E_{jn}|$$

   is well-defined, satisfies $\mu_0(\varnothing) = 0$, $\mu_0(A \cup B) = \mu_0(A) + \mu_0(B)$ if $A, B \in \mathcal{A}$ with $A \cap B = \varnothing$. [Can be done very similarly to constructing the product pre-measure, in class.]

   (d) Deduce that μ_0 is a pre-measure. Further deduce that μ_0 extends uniquely to a probability measure $\mu : \mathcal{B}(X) \to [0,1]$, where $\mathcal{B}(X)$ is the Borel σ-algebra on X. Finally, show that each "box set" $B = \prod_{i=1}^{\infty} E_i \in \mathcal{B}(X)$, with $\mu(B) = \lim_{n\to\infty} \frac{1}{2^n} \prod_{i=1}^{n} |E_i|$, [i.e. μ is the infinite product of normalized counting measure $\frac{1}{2}\gamma$ on $\{0,1\}$; write $\mu = (\frac{1}{2}\gamma)^{\times\mathbb{N}}$].

2. (Not every measure on a product space is a product measure.) Define $r : [0, 2\pi) \to \mathbb{R}^2$ by $r(t) = (\cos t, \sin t)$ so the image of r is a circle C. Define a measure $\rho : \mathcal{B}(\mathbb{R}^2) \to [0, \infty]$ by $\rho(E) = \lambda(r^{-1}(E \cap C))$. Verify that there exist no measures $\mu, \nu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ for which $\rho = \mu \times \nu$.

3. (Total variation for a complex measure.) Let $(X, \mathcal{M}, \nu)$ be a complex measure space. If $E \in \mathcal{M}$, a *finite $\mathcal{M}$-partition* of E is a finite sequence $E_1, \ldots, E_n$ in $\mathcal{M}$ for which $E_i \cap E_j = \varnothing$ for $i \neq j$ and $E = \bigcup_{j=1}^n E_j$.

   Show that

$$|\nu|(E) = \sup \left\{ \sum_{j=1}^{n} |\nu(E_j)| : E_1, \ldots, E_n \text{ is a finite } \mathcal{M}\text{-partition of } E \right\}$$

   defines a finite measure on $\mathcal{M}$. [Hint: define

$$\mu_1(E) = \sup \left\{ \sum_{j=1}^{\infty} |\nu(E_j)| : E_1, E_2, \ldots \text{ is a countable } \mathcal{M}\text{-partition of } E \right\}$$

$$\mu_2(E) = \sup \left\{ \left| \int_E f \, d\nu \right| : f \in M(X, \mathcal{M}), |f| \leq 1 \right\}.$$

   Show that μ_1 is σ-subadditive, μ_2 is σ-superadditive and $|\nu| \leq \mu_1 \leq \mu_2 \leq |\nu|$. For finiteness, recall in $\mathbb{C}$ that $|x + iy| \leq |x| + |y|$.]

4. (On absolute continuity.) On $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ define a measure $\nu(E) = \int_E \frac{dx}{|x|}$ (integral with respect to Lebesgue measure).

   (a) Verify that ν is σ-finite but neither locally finite, nor outer regular. [*Outer regularity* is the condition that for any E in $\mathcal{B}(\mathbb{R})$ that $\nu(E) = \inf\{\nu(U) : E \subseteq U,\, U \text{ open}\}$.]

   (b) Determine which of the following holds for the pair (ν, λ):

   (AC) $\nu \ll \lambda$, i.e. $\nu(E) = 0$ whenever $\lambda(E) = 0$;

   (AC') for every $\varepsilon > 0$ there is $\delta > 0$, so $\nu(E) < \varepsilon$ whenever $\lambda(E) < \delta$.

5. (Limitations of Radon-Nikodym Theorem.) Let $\gamma : \mathcal{B}([0, 1]) \to [0, \infty]$ denote the counting measure. Verify that $\lambda \ll \gamma$, but there is no Borel measurable $f : [0, 1] \to [0, \infty]$ for which $\lambda = f \cdot \gamma$ on $\mathcal{B}([0, 1])$.

6. Let $(X, \mathcal{M}, \nu)$ be a complex measure space, and $|\nu| : \mathcal{M} \to [0, \infty)$ its total variation, defined in Question 3.

   (a) Let $\mu : \mathcal{M} \to [0, \infty]$ be a σ-finite measure. Show that $\nu \ll \mu$ if and only if $|\nu| \ll \mu$. Moreover, $\frac{d|\nu|}{d\mu} = \left|\frac{d\nu}{d\mu}\right|$, μ-a.e.

   (b) Deduce that $\nu \ll |\nu|$ with $\left|\frac{d\nu}{d|\nu|}\right| = 1$, ν-a.e. [Why are $|\nu|$-null sets ν-null?]

7. (Conditional expectations.) Consider a probability space $(X, \mathcal{M}, \mu)$. Let $\mathcal{N} \subset \mathcal{M}$ be a σ-subalgebra. If $f \in L(\mu)$, then $(f \cdot \mu)|_{\mathcal{N}} \ll \mu|_{\mathcal{N}}$ (why?). Then we let

$$\mathbb{E}(f|\mathcal{N}) = \frac{d(f \cdot \mu)|_{\mathcal{N}}}{d\mu|_{\mathcal{N}}} \text{ in } L^1(X, \mathcal{N}, \mu|_{\mathcal{N}}) \subseteq L^1(N, \mathcal{M}, \mu).$$

   We let $(X, \mathcal{B}(X), \mu)$ denote the probability measure space from Question 1, and let $f \in L(\mu)$

   (a) Let

$$\mathcal{N}_n = \sigma\langle\{E_1 \times \cdots \times E_n \times \{0,1\} \times \{0,1\} \times \ldots\}\rangle \subset \mathcal{B}(X).$$

   Given f in $L(\mu)$, compute $\mathbb{E}(f|\mathcal{N}_n)$.

   (b) Show that

$$\lim_{n\to\infty} \int_{[0,1]} |f - \mathbb{E}(f|\mathcal{N}_n)|\, d\mu = 0.$$