PMATH 451/651, Winter 2018

Assignment #3 Due: Fri. Mar. 1 (new date).

For the purposes of this assignment let $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ denote the Lebesgue measure space.

1. If $f \in L(\lambda)$ we write $\int_{(a,b)} f\,d\lambda = \int_a^b f(t)\,dt$.

   (a) Compute $\displaystyle\int_0^\infty t^n e^{-st}\,dt$ where $s > 0$.

   [You may wish to differentiate $\int_0^\infty e^{-st}\,dt = \frac{1}{s}$; justify all steps.]

   (b) Let $J_0(t) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n (n!)^2} t^{2n}$. Show that $\displaystyle\int_0^\infty J_0(t) e^{-st}\,dt = \left(s^2+1\right)^{-1/2}$ for $s > 1$. You must justify any interchange of limit and integral.

   [You may use Newton's binomial series $(x+1)^\alpha = \sum_{n=0}^\infty \binom{\alpha}{n} x^n$ which is valid for $|x| < 1$, where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$.]

2. Let $f(x) = \frac{1}{\sqrt{x}} 1_{(0,1]}(x)$ for x in $\mathbb{R}$ and let $g(x) = \displaystyle\sum_{n=1}^\infty \frac{1}{2^n} f(x - r_n)$ where $\{r_n\}_{n=1}^\infty$ is an enumeration of $\mathbb{Q}$.

   (a) Show that $g \in \overline{L^+}(\lambda)$, hence $g < \infty$ λ-a.e.

   (b) Show that g is continuous at no point in $g^{-1}([0,\infty))$.

   (c) Show that $\int_{[a,b]} g^2\,d\lambda = \infty$ for $a < b$ in $\mathbb{R}$.

   Remark. There is no subinterval $[a,b]$ on which g is Riemann integrable.

3. **(a)** (Lusin's Theorem/Littlewood's Third Principle) Let $\mu : \mathcal{B}([a,b]) \to [0,\infty)$ be a finite measure and suppose $f : [a,b] \to \mathbb{R}$ is measurable. Given $\varepsilon > 0$, show that there is a compact set $K \subseteq [a,b]$ for which

   $$f|_K \text{ is continuous, and } \mu([a,b] \setminus K) < \varepsilon.$$

   [You probably want to use a few results from the lectures.]

   (b) How does one reconcile Lusin's Theorem with q. 2 (b), above?

4. (Integral is area under the graph.) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and $f \in L^+(\mu)$. Show that

$$G_f = \{(x, y) \in X \times [0, \infty) : y \leq f(x)\}.$$

Show that $G_f \in \mathcal{M} \otimes \mathcal{B}(\mathbb{R})$ and $\mu \times \lambda(G_f) = \int_X f \, d\mu$.

[Try this with a simple function, first.]

5. **(a)** Consider the measurable space $(X, \mathcal{B}) = ([0, 1], \mathcal{B}([0, 1]))$, and let γ the counting measure on $\mathcal{B}$. Let $D = \{(x, x) : x \in X\} \subset X \times X$. Show that the iterated integrals satisfy

$$\int_X \int_X 1_D(x, y) \, d\lambda(x) \, d\gamma(y) \neq \int_X \int_X 1_D(x, y) \, d\gamma(y) \, d\lambda(x).$$

Why does this not contradict Tonelli's theorem?

(b) Consider the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and let γ denote the counting measure. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be given by

$$f(m, n) = \begin{cases} 1 & \text{if } m = n \\ -1 & \text{if } n = m + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that the iterated integrals satisfy

$$\int_{\mathbb{N}} \int_{\mathbb{N}} f(m, n) \, d\gamma(m) \, d\gamma(n) \neq \int_{\mathbb{N}} \int_{\mathbb{N}} f(m, n) \, d\gamma(n) \, d\gamma(m).$$

Why does this not contradict Fubini's theorem?