PMATH 451/651, Winter 2018

Assignment #2 Due: Fri. Feb. 8

1. For this question, we consider the complete Lebesgue measure space $(\mathbb{R}, \mathcal{L}, \lambda)$. Our goal is to prove (e), below. In particular, $\mathcal{L} \subsetneq \mathcal{P}(\mathbb{R})$.

   (a) Show that if $E \in \mathcal{L}$ with $0 < \lambda(E) < \infty$ and $0 < \alpha < 1$, then there is some interval $I \subset \mathbb{R}$ for which $\lambda(E \cap I) > \alpha\lambda(I)$. [Hint: obtain $\sum_{i=1}^{\infty} \lambda(I_i) < \frac{1}{\alpha}\lambda(E)$ while $\lambda(E) \leq \sum_{i=1}^{\infty} \lambda(E \cap I_i)$.]

   (b) Show that if $E \in \mathcal{L}$ with $\lambda(E) > 0$, then $E - E = \{x - y : x, y \in E\}$ contains an interval.

   [Let $\alpha = \frac{3}{4}$ and I be as in (a). Show for $x \in (-\frac{\lambda(I)}{2}, \frac{\lambda(I)}{2})$ that $(x + E) \cap E \neq \varnothing$.]

   (c) Fix an irrational number ξ. Let $A = \{m + n\xi : m, n \in \mathbb{Z}\}$. Prove that A is dense in $\mathbb{R}$. [Dirichlet's pigeonhole principle.]

   (d) Consider the equivalence relation on $\mathbb{R}$ given by

   $$x \sim y \quad \Leftrightarrow \quad x - y \in A.$$

   Let $T \subset \mathbb{R}$ be a complete set of representatives, one from each equivalence class of $\sim$. [The existence of T requires axiom of choice.] Let $B = \{2m + n\xi : m, n \in \mathbb{Z}\}$ and $F = B + T$.

   Show that $F - F$ contains no intervals. Also show that $\mathbb{R} \setminus F = F + 1$, and hence $(\mathbb{R} \setminus F) - (\mathbb{R} \setminus F)$ contains no intervals as well.

   (e) Show that the set F from (d) satisfies the property that for every $S \in \mathcal{L}$ with $\lambda(S) > 0$, we have $S \cap F \notin \mathcal{L}$.

2. Construct a continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $F^{-1}(\mathcal{L}) \not\subseteq \mathcal{L}$.

   [Consider $x \mapsto \varphi(x) + x$, where φ is the Cantor staircase function. Why does this admit a continuosu inverse?]

   Recall: We call $f : \mathbb{R} \to \mathbb{R}$ *Lebesgue-measurable* if it is $\mathcal{L}$-$\mathcal{B}(\mathbb{R})$-measurable. The example above indicates why it is desirable the only pull-back Borel sets.

3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. If $h : X \to \mathbb{R}$ is measurable, define the *distribution measure* (also known by probabilists as the *distribution law*) of h by

$$\lambda_h : \mathcal{B}(\mathbb{R}) \to [0, \mu(X)] \text{ by } \lambda_h = \mu \circ h^{-1}.$$

(a) Show that λ_h is a finite Borel measure on $\mathbb{R}$, and hence is determined by its *distribution function* $F_h(x) = \lambda_h((-\infty, x])$. Also, show that $\mathbb{R} \setminus B$ is λ_h-null for any Borel set B containing the range $h(X)$.

(b) Show the following change of variables formula: if $f : \mathbb{R} \to [0, \infty)$ is Borel measurable, then

$$\int_X f(h(x))\, d\mu(x) = \int_{\mathbb{R}} f(t)\, d\lambda_h(t).$$

[Start with indicator functions $f = \chi_B$, then non-negative simple functions ...]

Notation. If $(X, \mathcal{M}, \mu)$ is a measure space, and $A \in \mathcal{M}$, we let the *restriction* of μ to A be given by $\mu|_A(E) = \mu(A \cap E)$ for E in $\mathcal{M}$. It is trivial to verify that $\mu|_A$ is a measure.

(c) Let $F : \mathbb{R} \to \mathbb{R}$ be non-decreasing continuous and bounded, consider the associated finite Borel measure $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_F)$, and let $\lambda_F = \mu_F \circ F^{-1}$, as above. Show that $\lambda_F|_{(F(-\infty),F(\infty))]} = \lambda|_{(F(-\infty),F(\infty))]}$, where $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is the Lebesgue measure.

(d) Let $H_x = 1_{[x,\infty)}$ and δ_x denote the Dirac measure at x. Compute $\lambda_x = \delta_x \circ H_x^{-1}$ as a Borel measure on $\mathbb{R}$. [How important is the continuity assumption in (c), above?]

4. Let φ denote the Cantor ternary function and $\mu = \mu_\varphi$ the associated measure. Let λ denote the Lebesgue measure. Compute each of the following.

(a) $\displaystyle\int_{[0,1]} \varphi(t)\, d\mu(t)$ **(b)** $\displaystyle\int_{[0,1]} \varphi(t)\, d\lambda(t)$ **(c)** $\displaystyle\int_{[0,1]} t\, d\mu(t)$