PMATH 451/651, Winter 2019

Assignment #1 Due: Jan. 25.

1. Let X be a set and $\{E_n\}_{n=1}^\infty$ a sequence of subsets of X. We denote

$$\limsup_{n\to\infty} E_n = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty E_k \text{ and } \liminf_{n\to\infty} E_n = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty E_k.$$

Hence $\limsup_{n\to\infty} E_n$ is the set of points which are in infinitley many of the sets E_n, while $\liminf_{n\to\infty} E_n$ is the set of points which are eventually in every E_n, for sufficiently large n. Suppose now that $(\mathcal{M}, \mu)$ is a measure on X and each E_n, above, is in $\mathcal{M}$.

(a) Show that $\mu(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} \mu(E_n)$.

(b) Show that $\mu(\limsup_{n\to\infty} E_n) \geq \limsup_{n\to\infty} \mu(E_n)$, provided that $\mu(\bigcup_{k=n}^\infty E_k) < \infty$ for some n.

(c) Create examples of sequences of sets $\{E_n\}_{n=1}^\infty$ and measure spaces $(X, \mathcal{M}, \mu)$ for which strict inequalities hold in (a) and (b), and for which the inequality of (b) fails when we drop the assumption that $\mu(\bigcup_{k=n}^\infty E_k) < \infty$ for some n. [This can be done with measures on $\mathcal{P}(\mathbb{N})$; or, if you prefer, with Lebesgue measure on $\mathbb{R}$.]

(d) Show that if $\sum_{n=1}^\infty \mu(E_n) < \infty$, then $\mu(\limsup_{n\to\infty} E_n) = 0$.

2. Let X be a set. For subsets $E, F \subset X$ we define their *symmetric difference* by

$$E \triangle F = (E \setminus F) \cup (F \setminus E).$$

Let $(\mathcal{M}, \mu)$ be a non-zero finite measure on X.

(a) Show that the relation on $\mathcal{M}$, $E \sim F \Leftrightarrow \mu(E \triangle F) = 0$, is an equivalence relation.

(b) Show that the function $\rho : \mathcal{M} \times \mathcal{M} \to [0, \infty]$, given by $\rho(E, F) = \mu(E \triangle F)$, satisfies $\rho(E, F) \leq \rho(E, G) + \rho(G, F)$. Deduce that ρ induces a metric d on the set of equivalence classes $M = \{[E] : E \in \mathcal{M}\}$: $d([E], [F]) = \rho(E, F)$.

(c) Show that the metric space (M, d), defined in (b) above, is complete (in the sense of metric spaces). [For a candidate limit for a Cauchy sequence, take a hint from Q. 1.]

3. Show that if a measure space $(X, \mathcal{M}, \mu)$ is semi-finite, then for any E in $\mathcal{M}$, $\mu(E) = \sup\{\mu(F) : F \subseteq E,\ F \in \mathcal{M} \text{ and } \mu(F) < \infty\}$.

   [Hint: if $\mu(E) = \infty$ it is best to proceed by a contradiction argument.]

4. Let $(X, \mathcal{A}, \mu_0)$ be a premeasure space, μ^* be the induced outer measure, $\mathcal{M}$ be the σ-algebra of μ^*-measurable sets, and $\mu = \mu^*|_{\mathcal{M}}$. For any family $\mathcal{E} \subset \mathcal{P}(X)$ we let

$$\mathcal{E}_\sigma = \left\{ E \subseteq X : E = \bigcup_{n=1}^{\infty} F_n \text{ for some } \{F_k\}_{k=1}^{\infty} \subseteq \mathcal{E} \right\},$$
$$\mathcal{E}_\delta = \left\{ E \subseteq X : E = \bigcap_{n=1}^{\infty} F_n \text{ for some } \{F_k\}_{k=1}^{\infty} \subseteq \mathcal{E} \right\}$$

   and $\mathcal{E}_{\sigma\delta} = (\mathcal{E}_\sigma)_\delta$.

   (a) Suppose $E \subseteq X$ satisfies $\mu^*(E) < \infty$. Show that there is $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq B$ and $\mu^*(E) = \mu^*(B)$.

   (b) Deduce that E as in (a) is μ^*-measurable if and only if there is $B \in \mathcal{A}_{\sigma\delta}$ such that $E \subseteq B$ and $\mu^*(B \setminus E) = 0$.

   (c) Show that if μ is σ-finite, then the equivalence of (b) remains true without the assumption that $\mu^*(E) < \infty$.

5. (Semifiniteness is neccessary for uniqueness of extension of a premeasure.) Let $\mathcal{A} \subset \mathcal{P}(\mathbb{Q})$ be the algebra generated by sets $\mathbb{Q} \cap (a, b]$, where $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.

   (a) Show that $\mu_0 : \mathcal{A} \to [0, \infty]$ given by

$$\mu_0(A) = \begin{cases} 0 & \text{if } A = \varnothing \\ \infty & \text{otherwise} \end{cases}$$

   is a premeasure on $\mathcal{A}$. Also, show that the σ-algebra generated by $\mathcal{A}$, $\sigma\langle\mathcal{A}\rangle$, is $\mathcal{P}(\mathbb{Q})$, and compute the measure μ on $\mathcal{P}(\mathbb{Q})$ arising from the cannonical pre-measure/outer measure/measure construction on $(\mathbb{Q}, \mathcal{A}, \mu_0)$.

   (b) Show that are infinitely many measures, $\nu : \mathcal{P}(\mathbb{Q}) \to [0, \infty]$ such that $\nu|_{\mathcal{A}} = \mu_0$.