Dual Space of L^1

Note. The main theorem of this note is on page 5. The secondary theorem, describing the dual of $L^\infty(\mu)$ is on page 8.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

We consider the subsets of X which are "μ-locally" in $\mathcal{M}$ as follows

$$\mathcal{M}_\mu = \{E \subseteq X : E \cap F \in \mathcal{M} \text{ for any } F \in \mathcal{M} \text{ with } \mu(F) < \infty\}.$$

Remarks. (i) It is easy to show that $\mathcal{M}_\mu$ is a σ-algebra, containg $\mathcal{M}$.

(ii) If μ is σ-finite, then $\mathcal{M}_\mu = \mathcal{M}$. [Easy exercise.]

Example: Let I be an uncountable set

$$\mathcal{C} = \{E \in \mathcal{P}(I) : \text{one of } E \text{ or } I \setminus E \text{ is countable}\}.$$

Let $\gamma : \mathcal{C} \to [0, \infty]$ denote the counting measure. Since finite sets with respect to γ are exactly the finite sets, we see that $\mathcal{C}_\gamma = \mathcal{P}(I)$.

Definition. An N in $\mathcal{M}_\mu$ be called *locally μ-null* if

$$\mu(N \cap F) = 0 \text{ whenever } F \in \mathcal{M} \text{ with } \mu(F) < \infty.$$

It is obvious that a countable union of locally μ-null sets is itself locally μ-null.

Definition. A function f in $\mathrm{M}(X, \mathcal{M}_\mu)$ is *(μ-)locally essentially bounded* if there is $\alpha > 0$ so $\{x \in X : |f(x) > \alpha\}$ is locally μ-null. Let

$$\|f\|_\infty = \inf\big\{M > 0 : \{x \in X : |f(x)| > M\} \text{ is locally } \mu\text{-null}\big\}.$$

We let $f \sim_{l.\mu} g$ if and only if $f = g$ locally μ-a.e. Let

$$L^\infty(\mu) = \{f \in \mathrm{M}(X, \mathcal{M}_\mu) : f \text{ is locally essentially bounded}\}/ \sim_{l.\mu} .$$

Notice that $f \sim_{l.\mu} 0$ if and only if $\|f\|_\infty = 0$, so $\|\cdot\|_\infty$ is well defined on $L^\infty(\mu)$. Recall that if $\mathcal{M}$ is σ-finite, $\mathcal{M}_\mu = \mathcal{M}$ and locally μ-null sets are μ-null.

Proposition. **(i)** *$\|\cdot\|_\infty$ is a norm on $L^\infty(\mu)$.*

(ii) *$L^\infty(\mu)$ is complete with respect to μ.*

Proof. (i) It is already observed that $\|f\|_\infty = 0$ only if $f = 0$, i.e. $f \sim_{l.\mu} 0$. If $f, g \in L^\infty(\mu)$ then for any $\varepsilon > 0$ we have

$$\begin{aligned}\{x \in X :& |f(x) + g(x)| > \|f\|_\infty + \|g\|_\infty + \varepsilon\} \\ &\subseteq \{x \in X : |f(x)| > \|f\|_\infty + \tfrac{\varepsilon}{2}\} \cup \{x \in X : |g(x)| > \|g\|_\infty + \tfrac{\varepsilon}{2}\}\end{aligned}$$

where the latter set is locally μ-null, so, using definition of essential supremeum, and taking ε to 0 we have $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. Likewise if $\alpha \in \mathbb{C}$ ith $\alpha \neq 0$ we have

$$\{x \in X : |\alpha f(x)| > |\alpha|\|f\|_\infty + \varepsilon\} = \{x \in X : |f(x)| > \|f\|_\infty + \tfrac{\varepsilon}{|\alpha|}\}$$

form which it follows that $\|\alpha f\|_\infty = |\alpha|\|f\|_\infty$.

(ii) Let us suppose $(f_n)_{n=1}^\infty$ in $L^\infty(\mu)$ satisfies $\sum_{n=1}^\infty \|f_n\|_\infty < \infty$. Given $\varepsilon > 0$ we have

$$\begin{aligned}\left\{x \in X : \sum_{n=1}^\infty |f_n(x)| > \sum_{n=1}^\infty \|f_n\|_\infty + \varepsilon\right\} \\ \subseteq \bigcup_{n=1}^\infty \{x \in X : |f_n(x)| > \|f_n\|_\infty + \tfrac{\varepsilon}{2^n}\}\end{aligned}$$

where the latter set is locally μ-null, so $f(x) = \sum_{n=1}^\infty f_n(x)$ is absolutely converging, hence makes sense for locally μ-a.e. x. Then

$$\begin{aligned}\left\{x \in X : \left|\sum_{n=1}^\infty f_n(x)\right| > \sum_{n=1}^\infty \|f_n\|_\infty + \varepsilon\right\} \\ \subseteq \bigcup_{n=1}^\infty \{x \in X : |f_n(x)| > \|f_n\|_\infty + \tfrac{\varepsilon}{2^n}\}\end{aligned}$$

is locally μ-null for any $\varepsilon > 0$, from which it follows that $\|\sum_{n=1}^\infty f_n\|_\infty < \sum_{n=1}^\infty \|f_n\|_\infty < \infty$. □

The following was observed, in the proof of the L^p-L^q duality.

Lemma. *If $f \in L^1(\mu)$, then there is a σ-finite (with respect to μ) subset E in $\mathcal{M}$ for which $f = f1_E$.*

Proof. We allow the usual abuse of notation, identifying f with a representative function.

Let $E_n = \{x \in X : |f(x)| > \frac{1}{n}\}$. Then $\mu(E_n) = \int 1_{E_n} \leq n \int |f| = n\|f\|_1 < \infty$, and $E = \{x \in X : |f(x)| > 0\} = \bigcup_{n=1}^{\infty} E_n$ is hence σ-finite. □

Remark. If we define $\bar{\mu} : \mathcal{M}_\mu \to [0, \infty]$ by

$$\bar{\mu}(E) = \begin{cases} \mu(E) & \text{if } E \in \mathcal{M} \\ \infty & \text{if } E \in \mathcal{M}_\mu \setminus \mathcal{M} \end{cases}$$

it is easy to verify that $\bar{\mu}$ is a measure, called the *saturation* of μ. Any σ-finite set in $\mathcal{M}_\mu$ is in $\mathcal{M}$. Hence, the above lemma tells us that $L^1(X, \mathcal{M}_\mu, \bar{\mu}) = L^1(X, \mathcal{M}, \mu)$.

Proposition. *Let $g \in L^\infty(\mu)$. Then the functional $\Phi_g : L^1(\mu) \to \mathbb{C}$ given by*

$$\Phi_g(f) = \int_X fg\,d\mu$$

is bounded with $\|\Phi_g\|_* = \|g\|_\infty$.

Proof. We shall use the usual abuse of notation and conflate equivalence classes of functions with functions themselves.

Let $f \in L^1(\mu)$. Since f admits a σ-finite supporting set E in $\mathcal{M}$, for which $f = f1_E$ we have that $|fg| \leq |f|\,\|g\|_\infty$ μ-a.e. Hence fg is integrable and

$$|\Phi_g(f)| \leq \int |fg| \leq \|g\|_\infty \int |f| = \|g\|_\infty \|f\|_1 .$$

Hence Φ_g is bounded with $\|\Phi_g\|_* \leq \|g\|_\infty$.

Given $\varepsilon > 0$, let $A_\varepsilon = \{x \in X : |g(x)| > \|g\|_\infty - \varepsilon\}$ which is in $\mathcal{M}_\mu$ Hence by definition of $\|g\|_\infty$ there is F in $\mathcal{M}$ with $\mu(F) < \infty$ for which $\mu(A_\varepsilon \cap F) > 0$. Let

$$f = \frac{1}{\mu(A_\varepsilon \cap F)} \overline{\mathrm{sgn} g}\, 1_{A_\varepsilon \cap F}.$$

Then we see that

$$\|f\|_1 = 1 \text{ so } \|\Phi_g\| \geq |\Phi_g(f)| = \int_{A_\varepsilon \cap F} |g| \geq \|g\|_\infty - \varepsilon.$$

It follows that $\|\Phi_g\|_* \geq \|g\|_\infty$. $\square$

We shall still require some approximation by finite sets to gain a duality result. The following definition very mildly generalizes that of Exercise 15 on p. 92 of *Real Analysis* by Folland; compare our condition (d) to his.

Definition. We say that μ is *decomposable* on $(X, \mathcal{M})$ if there is a family $\{F_i\}_{i \in I} \subset \mathcal{M}$ for which

(a) each $\mu(F_i) < \infty$,

(b) $F_i \cap F_j = \varnothing$ if $i \neq j$ and $\bigcup_{i \in I} F_i = X$,

(c) if E in $\mathcal{M}$ has $\mu(E) < \infty$, then

$$\mu(E) = \sum_{i \in I} \mu(E \cap F_i) := \sup \left\{ \sum_{i \in J} \mu(E \cap F_i) : J \subset I \text{ is finite} \right\}, \text{ and}$$

(d) if $E \subset X$ satisfies $E \cap F_i \in \mathcal{M}$ for each i, then $E \in \mathcal{M}_\mu$.

We shall refer to $\{F_i\}_{i \in I}$, above, as a *decomposition* for μ.

Remarks. **(i)** Obviously σ-finite implies decomposable (with countable decomposition).

(ii) Decomposable does not imply semifinite. (See A5, Q4; modify part (b).)

(iii) It is known that semifinite does not imply decomposable, but examples are too obscure to discuss.

(iv) If μ is decomposable with decomposition $\{F_i\}_{i \in I}$, then $N \in \mathcal{M}_\mu$ is locally μ-null if and only if $\mu(N \cap F_i) = 0$ for each i. Sufficiency follows from property (c) above, whereas necessity is obvious.

The definition of $L^\infty(\mu)$, comprising of $\mathcal{M}_\mu$-measurable functions, allows for the next result to hold very generally.

Theorem. *Suppose μ is decomposable on $(X, \mathcal{M})$. If $\Phi : L^1(\mu) \to \mathbb{C}$ is a bounded linear functional, then $\Phi = \Phi_g$ for some g in $L^\infty(\mu)$. Hence there is an isometric identification $L^1(\mu)^* \cong L^\infty(\mu)$.*

Proof. As with the proof of duality for $1 < p < \infty$, we shall break this into parts.

(I) Suppose that μ is finite.

As in the case $1 < p < \infty$, we have that if $E = \bigcup_{i=1}^\infty E_i$, $E_i \in \mathcal{M}$ and $E_i \cap E_j = \varnothing$ if $i \neq j$, then $1_E = \sum_{i=1}^\infty 1_{E_i} = \lim_{n\to\infty} \sum_{i=1}^n 1_{E_i}$ in $L^1(\mu)$. Then $\nu : \mathcal{M} \to \mathbb{C}$ given by

$$\nu(E) = \Phi(1_E)$$

satisfies $\nu(N) = 0$ if $N \in \mathcal{M}$, $\mu(N) = 0$ – in particular $\nu(\varnothing) = 0$ – and $\nu(E) = \sum_{i=1}^\infty \nu(E_i)$ for E as above. (Details are exactly as in the case $1 < p < \infty$.) Thus ν is a measure, $\nu \ll \mu$ so $\nu = g \cdot \mu$, where $g \in L^1(\mu)$.

Notice that since each element φ of $\mathcal{S} = \mathcal{S}(X, \mathcal{M})$ are bounded, $\varphi g \in L^1(\mu)$. Further, it is easily checked that $\Phi(\varphi) = \int \varphi g \, d\mu$. Hence

$$M_1(g) = \sup \left\{ \left| \int \varphi g \, d\mu \right| : \varphi \in \mathcal{S}, \|\varphi\|_1 \leq 1 \right\} \leq \|\Phi\|_*$$

For $\alpha > 0$, if $A_\alpha = \{x \in X : |g(x)| > \alpha\}$ satisfies $\mu(A_\alpha) > 0$. Then

$$f = \frac{1}{\mu(A_\alpha)} \overline{\operatorname{sgn} g} \, 1_{A_\alpha}$$

satisfies $\|f\|_1 = 1$ and hence

$$\alpha \leq \int fg = \frac{1}{\mu(A_\alpha)} \int_{A_\alpha} |g| = \left| \int fg \right| \leq M_\infty(g).$$

Using the definition of $\|g\|_\infty$ in this finite measure setting, we may take α up to $\|g\|_\infty$, we see that $\|g\|_\infty \leq M_\infty(g) < \infty$.

Now if $f \in L^1(\mu)$, there is a sequence $(\varphi_n)_{n=1}^\infty \subset \mathcal{S}$ for which $|\varphi_n| \leq |f|$ and $\lim_{n\to\infty} \varphi_n = f$. Then $|\varphi_n - f| \leq 2|f| \in L^1(\mu)$, and LDCT tells us that $\lim_{n\to\infty} \|\varphi_n - f\|_1 = 0$. Also $|\varphi_n g| \leq |fg| \in L^1(\mu)$ and $\lim_{n\to\infty} \varphi_n g = fg$

μ-a.e. (Notice that $\mathcal{M}_\mu = \mathcal{M}$ in this finite measure setting.) Thus continuity of Φ, and LDCT tells us that

$$\Phi(f) = \lim_{n\to\infty} \Phi(\varphi_n) = \lim \int \varphi_n g \, d\mu = \int fg d\mu$$

so $\Phi = \Phi_g$.

(II) Now suppose μ is decomposable.

Let $\{F_i\}_{i\in I}$ be a decomposition of μ. For each i let $L^q(F_i) = \{f \in L^q(\mu) : f1_{F_i} = f \text{ in } L^q(\mu)\}$ $(q = 1, \infty)$. Let $g_i \in L^\infty(F_i)$ be so $\Phi|_{L^1(F_i)} = \Phi_{g_i}$. Then let $g = \sum_{i\in I} g_i$, pointwise locally μ-a.e., i.e. $g(x) = g_i(x)$ for μ-a.e. x in F_i. We observe that

$$g^{-1}\big((\alpha,\infty)\big) = \bigcup_{i\in I} g_i^{-1}\big((\alpha,\infty)\big)$$

so $g \in M(X, \mathcal{M}_\mu)/\sim_{l.\mu}$, by part (d) of the definition of decomposability. Also the part (iv) of the last remark shows

$$\|g\|_\infty = \sup_{i\in I} \|g_i\|_\infty = \sup_{i\in I} \left\|\Phi|_{L^1(F_i)}\right\| \leq \|\Phi\| < \infty.$$

Recall that each f in $L^1(\mu)$ has a σ-finite support set E. Write $E = \bigcup_{n=1}^\infty E_n$ where each $E_n \in \mathcal{M}$ with $\mu(E_n) < \infty$. For each E_n, the set $I_{n,k} = \{i \in I : \mu(E_n \cap F_i) > \frac{1}{k}\}$ is finite, hence $I_n = \{i \in I : \mu(E_n \cap F_i) > 0\}$ is contable. Thus $I_E = \{i \in I : \mu(E \cap F_i) > 0\} = \bigcup_{n=1}^\infty I_n$ is countable; write $I_E = \{i_k\}_{k=1}^N$ where N is either finite or $N = \infty$. Thus

$$\Phi(f) = \sum_{i\in I_E} \Phi(f1_{F_i}) = \sum_{i\in I_E} \int fg_i = \sum_{k=1}^N \int fg_{i_k}.$$

If N is finite the sum above is $\int fg$. If $N = \infty$ then observe $|\sum_{k=1}^n \int fg_{i_k}| \leq |fg| \leq \|g\|_\infty |f|$ for each n. Hence by LDCT we have

$$\sum_{k=1}^\infty \int fg_{i_k} = \lim_{n\to\infty} \sum_{k=1}^n \int fg_{i_k} = \int \lim_{n\to\infty} \sum_{k=1}^n fg_{i_k} = \int fg.$$

Hence $\Phi = \Phi_g$ and we are done. □

Example. **(i)** Let $(I, \mathcal{C}, \gamma)$ be as in the example above where γ is counting measure. Recall that $\mathcal{C}_\gamma = \mathcal{P}(I)$ and $L^1(I, \mathcal{C}, \gamma) = L^1(I, \mathcal{C}_\gamma, \gamma)$.

Notice that $\{\{i\}\}_{i\in I}$ forms a decomposition for γ.

The theorem above gives that $L^1(\gamma)^* \cong L^\infty(\gamma)$.

(ii) Here is a non-semifinite, non-decomposable example.

Let $X = \{f, i\}$, and consider $\mu : \mathcal{P}(X) \to [0, \infty]$ be the unique measure satisfying $\mu(\{f\}) = 1$ and $\mu(\{i\}) = \infty$. Notice that $\{i\}$ is locally null. Indeed, $\varnothing \cap \{i\} = \varnothing = \{f\} \cap \{i\}$, so if $\mu(F) < \infty$, then $\mu(F \cap \{i\}) = 0$. Thus with our definition, $L^\infty(\mu) = \mathrm{span} 1_{\{f\}}$. It is easily checked that $L^1(\mu) = \mathrm{span} 1_{\{f\}}$, and, furthermore, that $L^1(\mu) \cong L^\infty(\mu)$.

Notice that the existence of the "infinite atom" $\{i\}$ means we cannot partition X into subsets of finite measure. Hence this example sits outside of the assumptions of the theorem, above.

We remark that if we insisted to take the usual definition of $L^\infty(\mu)$, i.e. that $f \in L^\infty(\mu)$ if and only if $\mu(\{x \in X : |f(x)| > \alpha\}) = 0$ for some $\alpha > 0$, then we would have that $L^\infty(\mu) = \mathrm{span}\{1_{\{f\}}, 1_{\{i\}}\}$ while still $L^1(\mu) = \mathrm{span} 1_{\{f\}}$. Hence the duality would fail: $L^1(\mu)^* \subsetneq L^\infty(\mu)$. This is further motivation for the definition for $L^\infty(\mu)$ as we have it, above, as opposed to the one in the present paragraph. □

Let me close this section by remarking that any example of a connected locally compact metric space is known to be σ-compact. (Conduct an internet search on this question for yourself.) Hence Radon measure on such spaces are σ-finite, thus decomposable.

I cannot come up with an example of a locally compact metric space which is not a union of open σ-compact sets. [E.g. my examples are of the form disjoint union of manifolds, or a locally compact group.] I might suggest that in all interesting cases, decomposability of Radon measures holds. Thus the theorem above is as close to global as one could conceivably need.

On the Dual Space of L^∞

Again, let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mathcal{M}_\mu$ and $L^\infty(\mu)$, as above. Proofs here will be more terse, since this material is beyond the curriculum for this class.

Lemma. *The space $\mathcal{S} = \mathcal{S}(X, \mathcal{M}_\mu)$ of $\mathcal{M}_\mu$-measurable simple functions (modulo equivalence locally μ-a.e.) is dense in $L^\infty(\mu)$.*

Proof. Given f in $L^\infty(\mu)$ and $\varepsilon > 0$, partition $\{z \in \mathbb{C} : |z| \leq \|f\|_\infty\} = \bigcup_{i=1}^n B_i$ into Borel subsets, each of diameter $\operatorname{diam}(B_i) < \varepsilon$. Let $A_i = f^{-1}(B_i) \in \mathcal{M}_\mu$. Then for any choice of points $b_i \in B_i$, $i = 1, \ldots, n$ we have $\|f - \sum_{i=1}^n b_i 1_{A_i}\|_\infty < \varepsilon$. $\square$

We now consider the space $A(\mathcal{M}_\mu, \mu)$ of functions $m : \mathcal{M}_\mu \to \mathbb{C}$ for which

- $m(N) = 0$ if N is locally μ-null – in particular $m(\varnothing) = 0$; and
- $m\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n m(E_i)$, if $E_i \cap E_j = \varnothing$ for $i \neq j$.

These are called *additive set functions* on $\mathcal{M}_\mu$, which are absolutely continuous with respect to μ. One may also wish to call these *finitely additive measures* (absolutely continuous with respect to μ) in obvious analogy to the definition of a complex measure. On should be warned, these do not act much like measures, though. For example, continuity from below may fail, an egregious violation of one of the key features we exploited in our integration theory. See the example at the end of this section.

Theorem. *Each bounded linear functional $\Phi : L^\infty(\mu) \to \mathbb{C}$ determines a unique element of $A(\mathcal{M}_\mu, \mu)$. Likewise, each element m of $A(\mathcal{M}_\mu, \mu)$ determines a bounded linear functional $\Phi : L^\infty(\mu) \to \mathbb{C}$ by*

$$\Phi(f) = \lim_{n\to\infty} \Phi(\varphi_n) \text{ where } (\varphi_n)_{n=1}^\infty \subset \mathcal{S}(X, \mathcal{M}_\mu)/\sim_{l.\mu}, L^\infty\text{-}\lim_{n\to\infty} \varphi_n = f$$

and for each $\varphi = \sum_{i=1}^n \alpha_i 1_{E_i}$ in $\mathcal{S}(X, \mathcal{M}_\mu)$ (standard form), we have

$$\Phi(\varphi) = \sum_{i=1}^n \alpha_i m(E_i). \qquad (\dagger)$$

Proof. The lemma above shows that and bounded linear functional $\Phi : L^\infty(\mu) \to \mathbb{C}$ is determined on $\mathcal{S} = \mathcal{S}(X, \mathcal{M}_\mu)/\sim_{l.\mu}$. But $m(E) = \Phi(1_E)$ then gives an element of $A(\mathcal{M}_\mu, \mu)$. It is clear that m is uniquely determined by Φ.

It is easy to show that given m in $A(\mathcal{M}_\mu, \mu)$, (†) determines a functional Φ_0 on $\mathcal{S}$ with

$$\|\Phi_0\| = \sup \left\{ \sum_{i=1}^{n} |m(E_i)| : E_1, \ldots, E_1 \text{ is an } \mathcal{M}_\mu\text{-measurable partition of } X \right\}.$$

(Compare with A4 Q4.) Hence if $\lim_{n\to\infty} \varphi_n = f = \lim_{n\to\infty} \psi_n$ for sequence form in $\mathcal{S}$, then $\lim_{n\to\infty} \Phi_0(\varphi_n) = \lim_{n\to\infty} \Phi_0(\psi_n)$, and this defines a unique value $\Phi(f)$. This, in turn defines a bounded linear functional on $L^\infty(\mu)$. □

There is an obvious embedding $L^1(\mu) \hookrightarrow A(\mathcal{M}_\mu, \mu)$. [Can you determine what this "obvious" embedding is?]

Example. Let us exhibit an element of $L^\infty(\mu)$ which is not in the range of the embedding of $L^1(\mu)$ implied above whenever $\mathcal{M}$ is infinite and decomposable.

A set $\mathcal{F} \subset \mathcal{M}_\mu$ is called a *μ-filter* if

- for any E, F in $\mathcal{F}$, $E \cap F$ is not locally μ-null, and
- if $F \in \mathcal{F}$, $F \subseteq E$, $E \in \mathcal{M}_\mu$, then $E \in \mathcal{F}$ too.

As an example, fix E in $\mathcal{M}_\mu$ non-locally-null and let $\mathcal{F}_E = \{F \in \mathcal{M} : E \subseteq F\}$.

A *μ-ultrafilter* is a μ-filter $\mathcal{U}$ for which if $\mathcal{U} \subseteq \mathcal{F}$ for another μ-filter $\mathcal{F}$, then $\mathcal{U} = \mathcal{F}$. Notice that if $E \in \mathcal{M}_\mu$, then

$$\text{either } E \in \mathcal{U} \text{ or } X \setminus E \in \mathcal{U}. \qquad (\heartsuit)$$

Indeed, if $E \cap F$ is non-locally-null for each F in $\mathcal{U}$ then $\{E\} \cup \mathcal{U}$ is also a filter, containing $\mathcal{U}$, whence $\{E\} \cup \mathcal{U} = \mathcal{U}$, i.e. $E \in \mathcal{U}$; otherwise $E \cap F$ is locally μ-null for some F in $\mathcal{U}$, whence $F \subseteq (X \setminus E) \cup (F \cap E)$ and thus for every other element F' of $\mathcal{U}$, $(X \setminus E) \cap F' \supseteq (F \cap F') \setminus (F \cap E)$ is not locally-null.

If $\mathcal{M}$ is infinite and decomposable then there exists an $\mathcal{M}$-partition $E_1, E_2, \dots$ of X into non-locally μ-null sets [why?], and we consider the μ-filter

$$\mathcal{F} = \left\{ F \in \mathcal{M} : \bigcup_{k=n}^{\infty} E_n \subseteq F \text{ for each } n \text{ in } \mathbb{N} \right\}.$$

Any chain Γ of μ-filters containing $\mathcal{F}$ has that $\bigcup_{E \in \mathcal{G}, \mathcal{G} \in \Gamma} E$ is also a filter. Hence by Zorn's lemma, a maximal μ-filter $\mathcal{U}$ containing $\mathcal{F}$ exists. It is evident that $\mathcal{U}$ is a μ-ultrafilter.

Let $\delta_{\mathcal{U}} : \mathcal{M}_\mu \to \mathbb{C}$ be given by

$$\delta_{\mathcal{U}}(E) = \begin{cases} 1 & \text{if } E \in \mathcal{U} \\ 0 & \text{if } X \setminus E \in \mathcal{U}. \end{cases}$$

An application of ($\heartsuit$) shows that $\delta_{\mathcal{U}} \in A(\mathcal{M}_\mu, \mu)$. However, $\delta_{\mathcal{U}}$ is not countable additive. Indeed, consider the sets $E_1, E_2, \dots$, above. No $E_n \in \mathcal{U}$, while $X = \bigcup_{i=1}^{\infty} E_i \in \mathcal{U}$ [why?]. Hence

$$1 = \delta_{\mathcal{U}}(X) \neq 0 = \sum_{n=1}^{\infty} \delta_{\mathcal{U}}(E_n).$$

Note further that

$$0 = \delta_{\mathcal{U}} \left(\bigcup_{i=1}^{n} E_i \right) < 1 = \delta_{\mathcal{U}}(X) = \delta_{\mathcal{U}} \left(\bigcup_{i=1}^{\infty} E_n \right)$$

so continuity from below fails.

It is known that axiom of choice (or at least an axiom strong enough to allow construction of ultrafilters) is required to construct elements of $L^\infty(\mu)^*$ which are <u>not</u> from the embedding of $L^1(\mu)$ into $A(\mathcal{M}_\mu, \mu)$, above. □

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