Dual Space of $C_0(X)$

We let (X,d) be a locally compact metric space.

Recall that a Borel measure $\mu : \mathcal{B}(X) \to [0,\infty]$ is *Radon* if it is

- locally finite [$\mu(K) < \infty$ for compact K],
- outer regular [$\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}$], and
- inner regular on open sets [$\mu(U) = \sup\{\mu(K) : K \subseteq U, K \text{ compact}\}$].

We saw in class that any Radon measure is in fact inner regular on all Borel sets finite for μ, hence on all sets σ-finite for μ. It may not be inner regular generally; see A5, Q4.

Proposition. *Let $\mu : \mathcal{B}(X) \to \mathbb{C}$ be a complex measure. Then*

$$|\nu| \text{ is Radon} \quad \Leftrightarrow \quad \text{each } \mathrm{Re}\nu^+, \mathrm{Re}\nu^-, \mathrm{Im}\nu^+, \mathrm{Im}\nu^- \text{ is Radon.}$$

Proof. We observe, for a finite positive measure μ, that μ is Radon if and only if for ash Borel E, there are $K \subseteq E \subseteq U$, K compact, U open, such that $\mu(U \setminus K) < \varepsilon$. Hence the inequalities

$$\mathrm{Re}\nu^+, \mathrm{Re}\nu^-, \mathrm{Im}\nu^+, \mathrm{Im}\nu^- \leq |\nu| \leq \mathrm{Re}\nu^+ + \mathrm{Re}\nu^- + \mathrm{Im}\nu^+ + \mathrm{Im}\nu^-$$

give the result immediately. $\square$

We will call a complex measure ν *Radon*, if it satisfies either of the equivalences of the proposition, above. We denote the set of all such measures by $M(X)$. It is straightforward to check that $M(X)$ is vector space (i.e. $(\alpha\nu + \rho)(E) = \alpha\nu(E) + \rho(E)$), and, moreover, that $|\nu|(X)$ defines a norm on $M(X)$. Indeed use any of the (implicit) definitions in A4 Q3 to verify

$$|\alpha\nu|(X) = |\alpha||\nu|(X),\ |\nu+\rho|(X) \leq |\nu|(X)+|\rho|(X) \text{ and } |\nu|(X) = 0 \ \Leftrightarrow\ \mu = 0.$$

Let for $f \in C_c(X)$

$$\|f\|_u = \sup\{|f(x)| : x \in X\}.$$

(The notation $\|f\|_\infty$ is also common, but we wish not to confuse the present notation withe the (local) essential supremum norm of L^∞.) We let

$$C_0(X) = \overline{C_c(X)}^{\|\cdot\|_u}.$$

We call these "functions vanishing at infinity". Being a closed subspace of $C_b(X) = \{f : X \to \mathbb{C} \,|\, f \text{ is continuous an bounded}\}$, $C_0(X)$ is complete with respect to $\|\cdot\|_u$. It is a simple exercise to show

$$f \in C_0(X) \quad \Leftrightarrow \quad \{x \in X : |f(x)| \geq \varepsilon\} \text{ is compact for each } \varepsilon > 0.$$

We shall not make direct use of this fact; it will be more important to use density by elements of $C_c(X)$ in $C_0(X)$. For purposes of the proof below we shall use the notations

$$C_0^{\mathbb{R}}(X) = \{f \in C_0(X) : f(X) \subset \mathbb{R}\},\ C_0^+(X) = \{f \in C_0(X) : f(X) \subset [0,\infty)\}.$$

Theorem. *Let $I : C_0(X) \to \mathbb{C}$ be a bounded linear functional, i.e. $\|I\| = \sup\{|I(f)| : \|f\|_u \leq 1\} < \infty$. Then there is a complex Radon measure $\nu : \mathcal{B}(X) \to \mathbb{C}$ such that $I(f) = \int_X f\,d\nu$. Furthermore, $\|I\| = |\nu|(X)$.*

Proof. (I) We will construct a "Jordan decomposition" of I, and hence obtain Radon measures.

Let $J = \mathrm{Re} \circ I|_{C_0^{\mathbb{R}}(X)}$, so $J : C_0^{\mathbb{R}}(X) \to \mathbb{R}$ is $\mathbb{R}$-linear. Let for f in $C_0^+(X)$

$$J^+(f) = \sup\{J(h) : h \in C_0^+(X), h \leq f\}.$$

Then for f, g in $C_0^+(X)$ and $c \geq 0$ we have

- $J^+(cf) = cJ^+(f)$ $[0 \leq h \leq f \Leftrightarrow 0 \leq ch \leq cf]$, and
- $J^+(f+g) = J^+(f) + J^+(g)$ $[0 \leq h_1 \leq f,\ 0 \leq h_2 \leq g \Rightarrow 0 \leq h_1 + h_2 \leq f + g$; if $0 \leq h \leq f + g$ then $h_1 = \max\{f, h\}$ and $h_2 = h - h_1$ satisfy $0 \leq h_1 \leq f$ and $0 \leq h_2 \leq g$ (as can be verified pointwise)$]$.

Now for f in $C_0^{\mathbb{R}}(X)$ let

$$J^+(f) = J^+(f^+) - J^+(f^-).$$

Just as with integrating real-valued functions, we find J^+ is real linear. For example, if $f, g \in C_0^{\mathbb{R}}(X)$, we have $(f+g)^+ - (f+g)^- = f + g = f^+ + g^+ - f^- - g^-$ and hence

$$\begin{aligned} J^+(f+g) &= J^+((f+g)^+) - J^+((f+g)^-) \\ &= J^+(f^+) + J^+(g^+) - J^+(f^-) - J^+(g^-) = J^+(f) + J^+(g). \end{aligned}$$

Scalar homogeneity is similar. Obviously

$$\|J^+\| = \sup\{|J^+(f)| : f \in C_0^{\mathbb{R}}(X), \|f\|_u \leq 1\} \leq \|I\| < \infty.$$

Now let $J^- = J - J^+$ and from the definition of J^+ is is immediate that J^- is positive, i.e. $J^-(C_0^+(X)) \subseteq [0, \infty)$. Also it is easy to see that $\|J^-\| \leq \|J^+\| + \|J\| = 2\|J\| < \infty$.

Write $I|_{C_0^{\mathbb{R}}(X)} = J + iK$ where $K = \mathrm{Im} \circ I|_{C_0^{\mathbb{R}}(X)}$. Again $K = K^+ - K^-$ where each $K^\pm$ is positive. We define for$L = J^\pm, K^\pm$ $\widetilde{L} : C_0(\mathbb{R}) \to \mathbb{C}$ by

$$\widetilde{L}(f) = L(\mathrm{Re} f) + iL(\mathrm{Im} f)$$

and we see that $\widetilde{L}$ is clearly $\mathbb{R}$-linear, and if $x = x + iy \in \mathbb{C}$ we have

$$\begin{aligned}\widetilde{L}(zf) &= \widetilde{L}(x\mathrm{Re} f - y\mathrm{Im} f + i[x\mathrm{Im} f + y\mathrm{Re} f]) \\ &= xL(\mathrm{Re} f) - yL(\mathrm{Im} f) + i[xL(\mathrm{Im} f) + yL(\mathrm{Re} f)] = z\widetilde{L}(f)\end{aligned}$$

and hence is $\mathbb{C}$-linear. Furthermore, each $\widetilde{J^+}, \ldots, \widetilde{K^-}$ are evidently positive.

(II) The Riesz Representation Theorem now provides measures $\mu_1, \ldots, \mu_4$ for which

$$\widetilde{J^+}(f) = \int_X f\, d\mu_1, \ldots, \widetilde{K^-}(f) = \int_X f\, d\mu_4 \qquad (\heartsuit)$$

for f in $C_c^{\mathbb{R}}(X)$. Then we have

$$\begin{aligned}\mu_1(X) &= \sup\{\mu_1(K) : K \subset X, K \text{ compact}\} \\ &\leq \sup\{J^+(f) : f \prec X\} \leq \left\|J^+\right\| < \infty\end{aligned}$$

and likewise for each μ_2, μ_3 and μ_4. Using ($\heartsuit$), we find for f in $C_c(X)$ that the measure $\nu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ satisifes

$$\begin{aligned}I(f) &= \widetilde{J^+}(f) - \widetilde{J^-}(f) + i\left[\widetilde{K^+}(f) - \widetilde{K^-}(f)\right] \\ &= \int f\, d\mu_1 - \int f\, d\mu_2 + i\left[\int f\, d\mu_3 - \int f\, d\mu_4\right] = \int f\, d\nu\end{aligned}$$

By continuity of I, and LDCT [if $\lim_{n\to\infty} \|f - f_n\|_u = 0$, use $(\sup_{n\in\mathbb{N}} \|f_n\|_u)1$ as a majorant], the equality above holds for f in $C_0(X)$, as well.

[Note: though we haven't proved it, it can be shown that $\nu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ is the Jordan decomposition of μ. We don't require this to finish the proof.]

(III) We show that $\|I\| = |\nu|(X)$.

First, we recall that $\nu \ll |\nu|$ and $|\frac{d\nu}{d|\nu|}| = 1$ $|\nu|$-a.e. Hence if $\|f\|_u \leq 1$ we have

$$\left|\int f\,d\nu\right| = \left|\int f\frac{d\nu}{d|\nu|}\,d|\nu|\right| \leq \int \left|f\frac{d\nu}{d|\nu|}\right|\,d|\nu| \leq |\nu|(X)$$

so $\|I\| \leq |\nu|(X)|$.

Conversely, we recall that $C_c(X)/\sim_\mu$ is dense in $L^1(\mu)$. Thus there is a sequence $(f_n)_{n=1}^\infty \subset C_c(X)$ for which $\lim_{n\to\infty} f_n = \frac{d\nu}{d|\nu|}$ in $L^1(|\nu|)$. However, $L^1(|\nu|)$ convergnece implies convergence in measure, which, in turn, implies that there is a subsequence for which $\lim_{k\to\infty} f_{n_k} = \frac{d\nu}{d|\nu|}$ ν-a.e. In particular, $\lim_{k\to\infty} |f_{n_k}| = 1$ ν-a.e., and we have

$$\lim_{k\to\infty} \frac{f_{n_k}}{\max\{|f_{n_k}|, 1\}} = \frac{d\nu}{d|\nu|} \quad \nu\text{-a.e.}$$

Hence if $\overline{g_k} = \frac{f_{n_k}}{\max\{|f_{n_k}|,1\}}$, then each $\|g_k\|_u \leq 1$, and by LDCT [with integrable majorant 1] we have

$$\|I\| \geq \lim_{k\to\infty} |I(g_k)| = \lim_{k\to\infty} \left|\int g_k\,d\nu\right| = \left|\int \overline{\frac{d\nu}{d|\nu|}}\,d\nu\right| = \int \left|\frac{d\nu}{d|\nu|}\right|^2 d|\nu| = |\nu|(X).$$

□

Written by Nico Spronk, for use by students of PMath 451/651 at University of Waterloo.