

Pure Math 352, Final Exam Information

Time: Saturday, December 12, 9:00-11:30 am.

Place: MC 2017.

Topics: All of the course: $\leq 1/3$ from topics prior to the midterm (\leq maximum modulus principle), $\geq 2/3$ topics from since the midterm. The exam will be roughly divided between

- definitions and theory from the lectures, summarised below;
- (possibly modified) questions from Assignments #1, #2 and #3, especially those which relate to, and motivate, ideas in the second half of the course; and
- (possibly modified) questions from Assignments #4, #5 and #6

List of Propositions, Lemmas and Theorems from Lectures

For any one of these results I may expect you to provide a proof. [Every theorem is based on a few simple ideas, some of which I indicate.]

- Midterm review sheet (statements and applications but *no proofs*)
- $\int_{\gamma} f(z)dz = 0$ for closed rectifiable $\gamma \Leftrightarrow$ primitive exists [(\Leftarrow) F.T.of C.; (\Rightarrow) $F(z) = \int_{\gamma_z} f(w)dw$.]
- Holomorphicity of continuous branches of logarithm; this generalises to other continuous inverse functions [$(f^{-1}(z) - f^{-1}(w))/(z - w) = 1/\{[f(f^{-1}(z)) - f(f^{-1}(w))]/(f(z) - f(w))\}$]
- L branch of logarithm $\Leftrightarrow L'(z) = 1/z$ & $e^{L(z_0)} = z_0$ [(\Leftarrow) $ze^{-L(z)}$ has zero derivative]
- Geometry of logarithm: $\text{Log}(\mathbb{C} \setminus (-\infty, 0]) = \{z : -\pi/2 < \text{Im } z < \pi/2\}$ (principal branch)
- Winding Lemma [$\gamma : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$ closed, pick disc D_0 about $\gamma(0)$, misses 0, then obtain analytic continuation of logarithm $\{(L_t, D_t)\}$ along γ in discs missing 0. Take imaginary part of $L_1(\gamma(1)) - L_0(\gamma(0))$ using $2\pi i$ -periodicity of exp. γ rectifiable, use F.T. of C.]
- γ closed, rectifiable: $\text{Ind}(\gamma, a) \in \mathbb{Z}$, 0 on unbounded component. [On components of $\mathbb{C} \setminus \gamma^*$, $z \mapsto \text{Ind}(\gamma, z)$ holomorphic by Morera's Theorem and Fubini's Theorem, see proof of Cauchy's Integral formula (homology version)]
- Cauchy's Integral formula (homology version) [statement only, proof too fussy]
- Cauchy's Theorem (homology version) [apply C.I.F.(h.v.) to $(z - z_0)f(z)$]
- Laurent series [build careful cycle in $A = A(z_0; r, R)$ so $\text{Ind}(\Gamma, a) = 0$ for $a \notin A$, manipulate geometric series, use uniform convergence on compacta to interchange limit and integral.]
- Residue Theorem [show that assumptions on curves and singularities means only finitely many singularities have non-zero index, then subtract suitable "small circles" cycle, apply C.T.(h.v.).]
- Methods of computing residues, simple poles, poles of order m .
- Argument principle [Residue Theorem: use Zero Lemma and Classification of Singularities Theorem to compute residues of f'/f at zeros and poles.]
- Rouché's Theorem [$||f(z)/g(z) + 1| < |f(z)/g(z)| + 1$ on γ^* implies f/g misses $[0, \infty)$, use F.T.of C. on suitable branch of log.]
- Open Mapping Theorem [$||f(z) - f(z_0)| \geq \delta > 0$ on $\partial D(z_0, r)$ implies $D(f(z_0), \delta) \subset f(D(z_0, r))$ by application of Rouché.]
- Non-vanishing derivative [proof fussy application of Rouché.]
- Local injectivity [nice convexity argument, then prove inverse continuous, then apply derivative argument just for logarithms, above.]
- Uniqueness of analytic continuation [largest t for which continuations coincide for $s \leq t$ must be 1.]