PMATH 352, FALL 2009

Assignment #6 Due: December 7

1. Use Rouché's Theorem to prove the Fundamental Theroem of Algebra: an nth degree polynomial with complex coefficients admits n zeros (counting multiplicity).

2. **(a)** Show that if f is holomorphic and non-constant on a neighbourhood of $\overline{D}(0,1)$ and $|f(z)| < 1$ for $z \in \overline{D}(0,1)$, there is a unique $z_0 \in D(0,1)$ such that $f(z_0) = z_0$. [Hint: Consider range of $f(z)/z$ for $z \in \partial D(0,1)$, in order to apply Rouché.]

   (b) Show that if f is continuous on $\overline{D}(0,1)$ and holomorphic on $D(0,1)$, and $|f(z)| \leq 1$ for $z \in \overline{D}(0,1)$, there is $z_0 \in \overline{D}(0,1)$ such that $f(z_0) = z_0$. [Hint: Let $r_n \to 1^-$; show, first, that there exists z_n so $f(r_n z_n) = z_n$.]

   BONUS: Either prove that z_0 in (b) is unique, or find a counterexample.

3. Determine the number of zeros of $f(z) = 1 + 6z^3 + 3z^{10} + z^{11}$ in the annulus $A(0; 1, 2)$. [Hint: An annulus is the difference of 2 discs.]

4. Let $\lambda > 1$. Prove that $f_\lambda(z) = \lambda - z - e^{-z}$ has a unique zero in the right half-plane $H = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. Moreover, show that this zero, x_λ, is real and that $\lim_{\lambda \to 1^+} x_\lambda = 0$.

5. (Principal branch of arctan) Let $S = \{z \in \mathbb{C} : -\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}$. Show that tan is injective on S and on no larger open set. Moreover, $\tan(S) = V = \mathbb{C} \setminus \big(i(-\infty, -1] \cup i[1, \infty)\big)$. Compute a formula for the inverse function $A = \operatorname{Arctan} : V \to S$.

   [Hint: Write $\tan z = i(-1 + \frac{2}{e^{i2z}+1})$ and decompose this into a composition of injective functions, starting from S and ending in V.]

6. Let $D_0 = D(1,1)$ and $f_0(z) = z^{1/2}$ denote the principal branch of square root.

   (a) Let $\gamma(t) = e^{i2\pi t}$ for $t \in [0,1]$. Find an analytic continuation $\{(f_t, D_t)\}_{t \in [0,1]}$ along γ, and show that $[f_1]_1 = [-f_0]_1$, i.e. in some neighbourhood of 1, $f_0(z) = -f_1(z)$.

   (b) Now let $\gamma(t) = e^{i4\pi t}$ for $t \in [0,1]$. Find an analytic continuation $\{(f_t, D_t)\}_{t \in [0,1]}$ along γ, and show that $[f_1]_1 = [f_0]_1$.

   [Compare these results with A4, Q4.]