PMATH 352, FALL 2009

Assignment #5 Due: November 27

1. **(a)** Let $f(z) = \dfrac{1}{z(z-1)(z-2)}$ for $z \in \mathbb{C} \setminus \{0,1,2\}$. Compute the Laurent series for f on each of the annuli $A(0;0,1)$, $A(0;1,2)$ and $A(1;0,1)$.

   [Don't be afraid of Cauchy products (A2, Q2), if you need them; but you should be nervous about taking the Cauchy product of 2 Laurent series. Cauchy products can be avoided.]

   (b) Let $\gamma(t) = \frac{3}{2}\cos(2t) + i6\sin(2t)$, $t \in [0, 2\pi]$. Use any reasonable method to compute $\mathrm{Ind}(\gamma, z)$ where $z \in \{0,1,2\}$.

   (c) Use any reasonable method to compute $\displaystyle\int_\gamma \frac{dz}{z(z-1)(z-2)}$.

2. **(a)** Let p, q be polynomials with complex coefficients for which q admits no roots on the real line, and $\deg q \geq \deg p + 2$. Show that if $f(z) = p(z)/q(z)$ then

$$\int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f, z_j)$$

   where $z_1, \ldots, z_k$ represent the zeros of q with $\mathrm{Im}\, z_j > 0$.

   (b) Compute $\displaystyle\int_0^\infty \frac{dx}{x^n+1}$, where $n \in \mathbb{N}, n \geq 2$.

   [Use the "wedge" $[0,R]\dot{+}\gamma_R|_{[0,2\pi/n]}\dot{+}[Re^{i2\pi/n}, 0]$ where $\gamma_R(t) = Re^{it}$.]

   (c) Compute, using any reasonable method

$$\int_0^\infty \frac{x^2-x}{x^6+1} dx, \qquad \int_0^\infty \frac{2x^2+1}{(x^4+1)(x^2+1)} dx.$$

3. Let $V \subset \mathbb{C}$ be open, $z_0 \in V$ and $f \in \mathcal{H}(V \setminus \{z_0\})$. If z_0 is a pole of f, z_0 is called *simple* if $\lim_{z \to z_0}(z - z_0)f(z)$ exists. (Equivalently, the principal part of the Laurent series of f about z_0 has only the term $\frac{c_{-1}}{z-z_0}$).

   (a) Let $\alpha < \beta$ in $\mathbb{R}$, z_0 be a simple pole of f, and $\gamma_r(t) = re^{it}$ for $r > 0$. Show that

$$\lim_{r \to 0^+} \int_{z_0+\gamma_r|_{[\alpha,\beta]}} f(z)dz = (\beta - \alpha) i \mathrm{Res}(f, z_0).$$

   This will allow us to compute some integrals by "sneaking around" simple poles.

   (b) Indicate why $\displaystyle\int_0^\infty \frac{\sin t}{t} dt$ exists and compute this integral.

[Note, first, that $\int_{\Gamma_{r,R}} \frac{e^{iz}}{z} dz = 0$ where

$$\Gamma_{r,R} = [r, R] \dot{+} \gamma_R|_{[0,\pi]} \dot{+} [-R, -r] \dot{-} \gamma_r| + [0, \pi]$$

for $0 < r < R$, and where $\gamma_\rho(t) = \rho e^{it}$ (Why?). Also verify each of the estimates

$$\left| \int_{\gamma_R|_{[0,\pi]}} \frac{e^{iz}}{z} dz \right| \leq \int_0^\pi |\exp(iRe^{it})| dt \leq 2\delta + \int_\delta^{\pi-\delta} e^{-R\sin t} dt$$

and use these to show that $\lim_{R\to\infty} \int_{\gamma_R|_{[0,\pi]}} \frac{e^{iz}}{z} dz = 0$.]

(c) Does $\displaystyle\int_{-\infty}^{\infty} \frac{e^{it}}{t} dt$ exist? Why is it that the above method cannot be applied directly to find $\displaystyle\int_{-R}^{R} \frac{\sin t}{t} dt$.

4. **(a)** Let p, q be polynomials with complex coefficients in two indeterminates such that such that $q(x, y) \neq 0$ for real numbers x, y with $x^2 + y^2 = 1$. Show that

$$\int_0^{2\pi} \frac{p(\cos t, \sin t)}{q(\cos t, \sin t)} dt = \int_{\partial D(0,1)} \frac{p\left(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z})\right)}{q\left(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z})\right)} \frac{dz}{iz}$$

(b) Compute, using (a) above, $\displaystyle\int_0^{2\pi} \frac{dt}{a + \sin t}$ where $a > 1$.

5. Let $f(z) = \frac{1}{\sin(1/z)}$ for $z \in \mathbb{C} \setminus \left(\{0\} \cup \left\{\pm\frac{1}{n\pi} : n \in \mathbb{Z} \setminus \{0\}\right\}\right)$.

(a) For each singularity of f, identify what type of singularity it is and compute the residue at that singularity.

(b) Compute $\displaystyle\int_{\partial D(0,r)} \frac{dz}{\sin(1/z)}$ for $r > 0$ with $r \neq \frac{1}{n\pi}$, $n \in \mathbb{N}$.

[Hint: You may not use the residue theorem (why?). However, note that $f(-z) = -f(z)$.]

(c) Compute $\displaystyle\int_{\partial D(0,r)} \frac{dz}{\sin(1/z^k)}$, $k \in \mathbb{N}$, for $r > 0$ with $r \neq \frac{1}{\sqrt[k]{n\pi}}$, $n \in \mathbb{N}$.

6. **(a)** Compute $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x^2 + 3x + 2}$. [Use the "keyhole" we saw in class.]

(b) Compute $\displaystyle\int_0^\infty \frac{\sqrt[n]{x}}{x^n + 1} dx$ where $n \in \mathbb{N}, n \geq 2$.

[Use principal branch, modify the wedge from 2 (b); avoid 0 (why?).]

(c) Compute $\displaystyle\int_0^\infty \frac{\ln x}{x^4 + 1} dx$.

[Try a branch cut $-i[0, \infty)$ for logarithm, i.e. $\text{Log}_i(z) = \ln|z| + i\arg_i(z)$, $-\frac{\pi}{2} < \arg_i(z) < \frac{3\pi}{2}$; and the curve $\Gamma_{r,R}$ from 3 (b), above.]