PMATH 352, FALL 2009

Assignment #4 Due: November 16

1. **(a)** Let $V \subset \mathbb{C}$ and $\Gamma(V)$ be the family of all closed curves in V. Show that the relation of homotopy on $\Gamma(V)$ is an equivalence relation, i.e. show for $\gamma_0, \gamma_1, \gamma_2$ in $\Gamma(V)$ that

$$\text{(reflexivity) } \gamma_0 \underset{V}{\sim} \gamma_0, \quad \text{(symmetry) } \gamma_0 \underset{V}{\sim} \gamma_1 \;\Rightarrow\; \gamma_1 \underset{V}{\sim} \gamma_0$$
$$\text{(transitivity) } \gamma_0 \underset{V}{\sim} \gamma_1,\, \gamma_1 \underset{V}{\sim} \gamma_2 \;\Rightarrow\; \gamma_0 \underset{V}{\sim} \gamma_2.$$

   (b) Show that a star-like region $V \subset \mathbb{C}$ is simply connected.

   (c) Prove that $[1, -1+i, -1-i] \underset{\mathbb{C}'}{\sim} \partial D(0,1)$, where $\mathbb{C}' = \mathbb{C} \setminus \{0\}$. Use this to compute $\int_{[1,-1+i,-1-i]} \frac{dz}{z} = 2\pi i$

2. **(a)** Let $\alpha : [0, \infty) \to \mathbb{C}$ be a continuous path for which $\alpha(0) = 0$, α is simple (i.e. $\alpha(s) = \alpha(t)$ only if $s = t$), and $\lim_{t\to\infty} |\alpha(t)| = \infty$. Prove for any closed curve γ with $\gamma^* \subset V_\alpha$, and $a \in \alpha^*$ that $\text{Ind}(\gamma, a) = 0$, and hence V_α admits a branch of logarithm.

   [Hint: α^* is connected and unbounded.]

   BONUS: **(a')** Show that V_α is simply connected. **(a")** Show that V_α is connected.

   (b) Let $\alpha(t) = te^{it}$. Let for $z \in V_\alpha$, θ_z be the unique element in $(0, 2\pi)$ for which $|z|e^{i|z|} = ze^{i\theta_z}$. Compute, in terms of θ_z, an explicit formula for the unique branch of logarithm $\text{Log}_\alpha \in \mathcal{H}(V_\alpha)$ for which $\text{Log}_\alpha(1) = 0$.

   (c) Produce a clearly labelled sketch of the region $\text{Log}_\alpha(V_\alpha)$ in $\mathbb{C}$.

3. Let for $\alpha \in \mathbb{C} \setminus \{0\}$, $z \mapsto z^\alpha$ be the principal branch of the power function with domain $V_1 = \mathbb{C} \setminus (-\infty, 0]$.

   (a) Compute the following: $(-i)^{1/2}$, i^i, 5^{i+1}.

   (b) Let $\alpha > 0$. Determine all points $t_0 \in (-\infty, 0]$, for which $\lim\limits_{z \mapsto t_0, z \in V_1} z^\alpha$ exists.

   (c) Indicate the range of the function $z \mapsto z^i$. Illustrate with a diagram. Be sure to include the image of a typical circle $\partial D(0,r) \setminus \{-r\}$ $(r > 0)$ and a typical ray $(0, \infty)\zeta$ $(\zeta \in \mathbb{T} \setminus \{-1\})$.

4. **(a)** Show, by way of diagrams, that the squaring function $z \mapsto z^2$ on $\mathbb{C}$ takes each region $S_0 = \{z : \operatorname{Re} z > 0\} \cup i(0,\infty)$ or $S_1 = \{z : \operatorname{Re} z < 0\} \cup i(-\infty, 0)$ onto the punctured plane $\mathbb{C}' = \mathbb{C} \setminus \{0\}$. Be sure to indicate the image of a typical circle $\partial D(0,r) \cap S_j$ $(r > 0)$, and of a typical ray $(0,\infty)\zeta$ $(\zeta \in \mathbb{T} \cap S_j)$, for $j = 0, 1$.

Use this to show that the Riemann surface $\mathcal{R}_{\sqrt{\cdot}}$ is two copies of the punctured plane $\mathbb{C}'_{(0)} \sqcup \mathbb{C}'_{(1)}$ with topology described as follows:

- if $z \in (\mathbb{C} \setminus (-\infty, 0])_{(j)}$ $(j = 0, 1)$ then z admits ordinary neighbourhoods $D(z,r)_{(j)}$ provided $0 < r < \operatorname{dist}(z, (-\infty, 0]_{(j)})$;
- if $z \in (-\infty, 0]_{(0)}$ then neighbourhoods are of the form $\{z \in D(z,r)_{(0)} : \operatorname{Im} z \geq 0\} \sqcup \{z \in D(z,r)_{(1)} : \operatorname{Im} z < 0\}$ provided $0 < r < |z|$; and
- if $z \in (-\infty, 0]_{(1)}$ then neighbourhoods are of the form $\{z \in D(z,r)_{(1)} : \operatorname{Im} z \geq 0\} \sqcup \{z \in D(z,r)_{(0)} : \operatorname{Im} z < 0\}$ provided $0 < r < |z|$.

(b) Formulate a description of the Riemann surface for the nth-root function, $\mathcal{R}_{\sqrt[n]{\cdot}}$, $n \in \mathbb{N}$, $n \geq 2$. (No proof is required.)

BONUS: Formulate a description of $\mathcal{R}_{(\cdot)^{n/m}}$ where $n, m \in \mathbb{N}$, $m > 1$, $\gcd(n, m) = 1$.