PMATH 352, FALL 2009

Assignment #3 Due: October 28

1. (More integrals.) Evaluate

   (a) $\displaystyle\int_{\partial D(0,r)} \frac{\sin z}{z} dz \quad (r > 0)$ **(b)** $\displaystyle\int_{\partial D(0,2)} \frac{\cos z}{1+z^2} dz$

   (c) $\displaystyle\int_{\partial D(0,r)} \frac{z^2+1}{z(z^2+4)} dz \quad 0 < r < 2 \text{ and } r > 2.$

   [Don't be afraid to use all tools at your disposal including uniform convergence, Mr. Cauchy, and partial fractions.]

2. (More series.) Show that the function $\sec = \frac{1}{\cos}$ admits power series about 0 of the form

   $$\sec z = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} z^{2n}$$

   where the constants E_{2n}, called *Euler's numbers*, satisfy the recurrence $E_0 = 1$,

   $$E_{2n} - \binom{2n}{2n-2} E_{2n-2} + \binom{2n}{2n-4} E_{2n-4} + \dots (-1)^{n-1}\binom{2n}{2} E_2 + (-1)^n = 0.$$

   BONUS: Show that

   $$\lim_{n\to\infty} \frac{(2n+2)(2n+1)E_{2n}}{E_{2n+2}} = \left(\frac{\pi}{2}\right)^2,$$

   i.e. establish that the limit exists.

3. Show that if $f \in \mathcal{H}(\mathbb{C})$ with $f(t) = e^t$ for each t in $\mathbb{R}$, then $f(z) = e^z$ on $\mathbb{C}$.

   [Hint: Entire functions admit power series with infinite radii of convergence; or use Zero Lemma.]

4. **(a)** Suppose that $f \in \mathcal{H}(\mathbb{C})$ and there is a constant $C > 0$ such that $|f(z)| \le Ce^{\operatorname{Re} z}$ for all z. Prove that there is a constant c in $\mathbb{C}$ such that $f(z) = ce^z$.

   (b) Suppose that $f \in \mathcal{H}(\mathbb{C})$ satisfies $|f(z)| \le C(1+|z|)^n$ for all z, for some $C > 0$ and n in $\mathbb{N}$, if and only if f is a polynomial.

   [Hint: Liouville Slugger.]

5. **(a)** Suppose that $f : \overline{D}(0,1) \to \mathbb{C}$ is continuous and holomorphic on $D(0,1)$ with $f(0) = 0$ and $|f(z)| \le C$ on $\overline{D}(0,1)$. Show that $|f(z)| \le C|z|$ for z in $\overline{D}(0,1)$.

   (b) Suppose now only that $f \in \mathcal{H}(D(0,1))$ with $f(0) = 0$ and $|f(z)| \le C$ on $D(0,1)$. Show that $|f(z)| \le C|z|$ for z in $D(0,1)$

   [Hint: Consider $z \mapsto f(z)/z$ in (a); $z \mapsto f(rz)$, $0 < r < 1$, in (b).]

6. **(a)** Suppose $V \in \mathbb{C}$ is open, $f, g \in \mathcal{H}(V)$ and z_0 is a zero of order n and m respectively, for f and g, where $1 \leq m \leq n$. Show that f/g has a removable singularity at z_0.

   (b) Suppose $f \in \mathcal{H}(\mathbb{C})$ with $f(n) = 0$ for each n in $\mathbb{Z}$. Show that each singularity of $f(z)/\sin(\pi z)$ is removable.

   (c) (An abstract characterization of sine.) Suppose that f in $\mathcal{H}(\mathbb{C})$ satisfies that $f(z+1) = -f(z)$ (1 *skew-periodicity*), $f(0) = 0$ and $|f(z)| \leq Ce^{\pi|\mathrm{Im}\, z|}$ on $\mathbb{C}$, for some $C > 0$. Then $f(z) = c\sin(\pi z)$ for some constant c.