PMATH 352, FALL 2009

Assignment #2 Due: October 14

1. If $V \subset \mathbb{C}$ is open, $f : V \to \mathbb{C}$, $z \in V$ and $\zeta \in \mathbb{T}$, i.e. $\zeta \in \mathbb{C}$ with $|\zeta| = 1$, let the *directional derivative* be given by

$$D_\zeta f(z) = \lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \frac{f(z+t\zeta) - f(z)}{t}$$

when the limit exists.

(a) Show that if f is $\mathbb{C}$-differentiable at z then $D_\zeta f(z) = f'(z)\zeta$ for each ζ in $\mathbb{T}$.

Remark: If $f'(z) \neq 0$ then one can interpret for ζ, ξ in $\mathbb{T}$, the ratio $D_\zeta f(z)/D_\xi f(z)$, which by (a) is $\zeta\bar{\xi}$, as the angle made by the curves $t \mapsto f(z+t\zeta)$ and $t \mapsto f(z+t\xi)$. Thus it is often said that "f preserves angles at z", or "f is *conformal* at z".

(b) Use (a) to deduce the Cauchy-Riemann equations for f at z.

(c) Prove or provide a counter-example to the following statement: If there is an a in $\mathbb{C}$ such that $D_\zeta f(z) = a\zeta$ for each ζ in $\mathbb{T}$, then f is $\mathbb{C}$-differentiable at z.

2. **(a)** Let $f(z) = \sum_{n=0}^\infty c_n(z-z_0)^n, g(z) = \sum_{n=0}^\infty d_n(z-z_0)^n$ be power series about z_0 with respective radii of convergence R, S. Compute a power series expansion for fg about z_0 and show that its radius of convergence is at least $\min\{R,S\}$. BONUS: Prove that the radius of convergence is $\min\{R,S\}$, or provide a counterexample to show otherwise.

(b) Prove that for w, z in $\mathbb{C}$ that $\exp(w+z) = \exp(w)\exp(z)$. Use this to deduce the fomulas

$$\begin{aligned}\cos(w+z) &= \cos(w)\cos(z) - \sin(w)\sin(z), \text{ and}\\ \sin(w+z) &= \cos(w)\sin(z) + \sin(w)\cos(z).\end{aligned}$$

3. Devise a power series $f(z) = \sum_{n=0}^\infty c_n z^n$ with radius of convergence 1 such that

(i) $f(z) = \sum_{n=0}^\infty c_n z^n$ converges uniformly on $\overline{D}(0,1)$, and

(ii) the power series describing $f''(z)$ converges at no point on $\mathbb{T} = \partial D(0,1)$.

4. **(a)** Prove the *Abel summation formula* for sequences $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty \subset \mathbb{C}$:

$$\sum_{k=0}^n a_k b_k = a_n \sum_{k=0}^n b_k - \sum_{k=0}^{n-1}\sum_{j=0}^k b_j(a_{k+1} - a_k).$$

[Use induction starting at $n = 1$.]

(b) Suppose $\sum_{k=0}^{\infty} c_k z^k$ has radius of convergence R, $|z_0| = R$ and $\sum_{k=0}^{\infty} c_k z_0^k$ converges. Prove that

$$\lim_{\substack{t\to 1\\ t\in[0,1)}} \sum_{k=0}^{\infty} c_k (tz_0)^k = \sum_{k=0}^{\infty} c_k z_0^k.$$

[Hint: Establish that the series converges uniformly on $[0, z_0]$.]

5. Given a sequence of distinct points $a_1, \ldots, a_n$ in $\mathbb{C}$, we denote by $[a_1, \ldots, a_n]$ the <u>closed</u> chain $[a_1, a_2] \dotplus \ldots \dotplus [a_{n-1}, a_n] \dotplus [a_n, a_1]$. Compute each of the following chain integrals

   (a) $\displaystyle\int_{[1,-1+i,-1-i]} \frac{dz}{z}$ **(b)** $\displaystyle\int_{[i,-1,-i,1]} \frac{dz}{z}$ **(c)** $\displaystyle\int_{[1+\frac{i}{2},-1-i,-1+i,1-\frac{i}{2},-\frac{1}{2}]} \frac{dz}{z}.$

   [You will save yourself a headache by first devising a formula for $\int_{[a,b]} \frac{dz}{z}$, provided $0 \notin [a, b]$.]

6. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

   $$f(t) = \frac{1}{1+t^2}.$$

   Show that f is $\mathbb{R}$-analytic in the sense that $f(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$, with all a_n in $\mathbb{R}$, for t in $(t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$, for every t_0 in $\mathbb{R}$. Calculate, for every t_0, the radius of convergence for the power series describing f about t_0.

   [Hint: This may really be a complex analysis question.]

7. **(a)** Let $I \subset \mathbb{R}$ be an open interval and $w : [a, b] \times I \to \mathbb{R}$ be continuous on its domain, and continuously partially differentiable in the second variable. Let $g(x) = \int_a^b w(t, x)dt$ for x in I. Prove that g is continuously differentiable on I with $g'(x) = \int_a^b \frac{\partial w}{\partial x}(t, x)dt$.

   [Hint: MVT tells you $w(t, x+h) - w(t, x) = \frac{\partial w}{\partial x}(t, x + h_t)h$; you may use, without proof, that w is uniformly continuous on compact subsets of $[a, b] \times I$.]

   (b) Let $V \subset \mathbb{C}$ be open and $\varphi : [a, b] \times V \to \mathbb{C}$ be continuous on its domain, and holomorphic in the second variable, i.e. $u = \operatorname{Re}\varphi$ and $v = \operatorname{Im}\varphi$ are continuously partially differentiable in the variables $x = \operatorname{Re} z$, $y = \operatorname{Im} z$ and satisfy the Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Prove that $g(z) = \int_a^b \varphi(t, z)dt$ is holomorphic on V with $g'(z) = \int_a^b \frac{\partial \varphi}{\partial z}(t, z)dt$ where $\frac{\partial \varphi}{\partial z}(t, z) = \lim_{h\to 0} \frac{\varphi(t,z+h)-\varphi(t,z)}{h}$. [Warning: Holomorphic functions admit no straighforward analogue of MVT.]

   (c) Use Cauchy's integral formula and (b) to deduce Cauchy's integral formula for derivatives: if $f : V \to \mathbb{C}$ is holomorphic and $\overline{D}(z, r) \subset V$, then

   $$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\partial D(z,r)} \frac{f(w)}{(w-z)^{n+1}} dw.$$