PMATH 352, FALL 2009

Assignment #1 Due: September 25

1. **(a)** Let ζ_{16} be the 16th primitive root of unity. Explicitly compute ζ_{16} in Euclidean form, in terms of radicals; i.e. find, α, β in $\mathbb{R}$ such that $\zeta_{16} = \alpha + i\beta$ and each α and β can be expressed in terms of integers and radicals.

   (b) Compute all the solutions, in $\mathbb{C}$, of the equation $z^6 = -\frac{27}{2} + i\frac{27\sqrt{3}}{2}$. Prove (without appealing to the "Fundamental Theorem of Algebra", which we haven't covered in this class, yet) that your list is complete.

   [I would not recommend trying to derive formulas in terms of radicals. One such formula, which you may find on the internet, is $\sqrt{3}((1+i\sqrt{3})^{1/3} + (1-i\sqrt{3})^{1/3})/2^{4/3}$. To understand this you must comprehend the meaning of "$z^{1/3}$", which requires the notion of "principle branch of logarithm", which we shall cover later.]

   (c) Let $n > 1$ be a fixed natural number and $\zeta = \zeta_n$ the nth primitive root of unity. Prove that
   $$\sum_{k=0}^{n-1} \zeta^k = 0.$$

2. Let $V \subset \mathbb{C}$ be open and $z \in \mathbb{C}$.

   (a) If $f : V \to \mathbb{C}$ is $\mathbb{C}$-differentiable at z, show that f is continuous at z.

   (b) Prove the *product rule*: if $f, g : V \to \mathbb{C}$ are each $\mathbb{C}$-differentiable at z, then so too is fg with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

3. Let $V \subset \mathbb{C}$ be open and $z \in \mathbb{C}$.

   (a) Show that $f : V \to \mathbb{C}$ is differentiable at z if and only if there is a in $\mathbb{C}$ and a function $E : D(0,r) \to \mathbb{C}$ (where $r > 0$ is such that $D(z,r) \subset V$) such that
   $$f(z+h) = f(z) + ah + E(h) \text{ for } h \in D(0,r),$$
   $$E(0) = 0 \text{ and } \lim_{h\to 0} \frac{E(h)}{h} = 0.$$

   (b) Prove that for the "error" function E, in (a), above, that for any $\varepsilon > 0$ there is $\delta > 0$ such that for $|h| < \delta$ we have $|E(h)| \leq \varepsilon|h|$.

   (c) Prove the *chain rule*: If $f : V \to \mathbb{C}$ is differentiable at z, U is an open set containing $f(z)$ and $g : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable at $f(z)$, then the composition $g \circ f$ is $\mathbb{C}$-differentiable at z with
   $$(g \circ f)'(z) = g'\big(f(z)\big)g'(z).$$

4. **(a)** Let $n \in \mathbb{N}$ and $p : \mathbb{C} \to \mathbb{C}$ be given by $p(z) = z^n$. Show that the $\mathbb{C}$-derivative $p'(z)$ exists for each z and is equal to nz^{n-1}.

   (b) Let $q : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be given by $q(z) = \frac{1}{z}$. Show that $q'(z)$ exists for each $z \neq 0$ and is equal to $-\frac{1}{z^2}$. Deduce from this, and from the differentiation rules above, the *quotient rule*: if $f, g : V \to \mathbb{C}$ are $\mathbb{C}$-differentiable at z in V and $g(z) \neq 0$, then f/g is $\mathbb{C}$-differentiable at z with

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

5. Display an example of a function $f = u + iv$ ($u = \operatorname{Re} f$, $v = \operatorname{Im} f$), defined on $\mathbb{C}$, such that:

   (a) f is $\mathbb{C}$-differentiable at 0, the partial derivatives u_x, u_y exist in a neighbourhood of 0, but at least one of the partial derivatives u_x, u_y fails to be continuous at 0.

   (b) f is $\mathbb{C}$-differentiable at 0, but at no other point in $\mathbb{C}$.

   (c) The Cauchy-Riemann equations are satisfied at 0, f is continuous at 0, but f is not $\mathbb{C}$-differentiable at 0.

   Of course each example must contain clear reasoning as to why it exhibits the intended properties.

6. Given a sequence $(c_n)_{n=1}^\infty \subset \mathbb{C}$ define

$$R = \sup\left\{r \geq 0 : (r^n c_n)_{n=1}^\infty \text{ is bounded in } \mathbb{C}\right\}.$$

   (a) Prove that if $R > 0$ then

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \begin{cases} 1/R & \text{if } R < \infty \\ 0 & \text{if } R = \infty \end{cases}$$

   Also show that, $R = 0$ if and only if the sequence $(\sqrt[n]{|c_n|})_{n=1}^\infty$ is unbounded.

   (b) Suppose, morerover, that $L = \lim_{n\to\infty} |c_{n+1}|/|c_n|$ exists. Then

$$L = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \begin{cases} 1/R & \text{if } R < \infty \\ 0 & \text{if } R = \infty. \end{cases}$$

   [For (b), first show that if $r < 1/L$ ($r > 0$ in the case that $L = 0$) then there is N in $\mathbb{N}$, such that for $n \geq N$, $r^{n+1}|c_{n+1}| < r^n|c_n|$ and deduce that $(r^n c_n)_{n=1}^\infty$ is bounded.]