MATH 351, FALL 2017

Assignment #7 Not for submission.

1. Given a compact metric space (X, d_X) and a point x_0 in X let

$$C_{x_0}(X) = \{f \in C(X) : f(x_0) = 0\}$$

which is a uniformly closed subspace.

(a) Let (Y, d_Y) be another compact metric space, and $\psi : Y \to X$ be a continuous bijection and y_0 in Y satisfies $\psi(y_0) = x_0$. Show that $\Psi : C(Y) \to C(X)$ given by $\Psi(f) = f \circ \psi$ is

- an isometry;
- satisfies $\Psi(C_{y_0}(Y)) = C_{x_0}(X)$; and
- is linear and satisfies satisfies $\Psi(fg) = \Psi(f)\psi(g)$ (pointwise multiplication).

(b) Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Show that the maps

$$\psi_1 : \mathbb{R} \to (-1, 1),\ \psi_1(t) = \frac{t}{|t| + 1}$$
$$\psi_2 : (-1, 1) \to S \setminus \{(-1, 0)\},\ \psi_2(s) = (\cos(\pi s), \sin(\pi s))$$

are *homeomorphisms*, i.e. continuous and bijective with continuous inverse.

(c) Let $C_\infty(\mathbb{R}) = \{f \in C(\mathbb{R}) : \lim_{t\to\pm\infty} f(t) = 0\}$. Show that $C_\infty(\mathbb{R})$ is an algebra and closed algebra of $(C_b(\mathbb{R}), \|\cdot\|_\infty)$.

(d) Let $\psi = \psi_2 \circ \psi_1 : \mathbb{R} \to S \setminus \{(-1, 0)\}$. Show that the map $\Psi : C_{(-1,0)}(S) \to C_\infty(\mathbb{R})$ given by $\Psi(f) = f \circ \psi$ is an isometry, is linear and satisfies $\Psi(fg) = \Psi(f)\Psi(g)$ for $f, g \in C_{(-1,0)}(S)$.

(e) Let $A = \{t \mapsto p(t)q(e^{-t^2}) : p, q$ are polynomials with$q(0) = 0\} \subset C_\infty(\mathbb{R})$. Show that A is uniformly dense in $C_\infty(\mathbb{R})$.

2. Let $M > 0$ and

$$D_{1,M} = \left\{ f \in C[0,1] : \|f\|_\infty \leq 1,\ f' \text{ exists on } (0,1) \text{ with } \sup_{t\in(0,1)} |f'(t)| \leq M \right\}.$$

Is it possible, for some M that $D_{1,M}$ is dense in the ball $B[0,1]$ in $(C[0,1], \|\cdot\}_\infty)$?

[There are two distinctive ways to think about this: BCT and AAT.]

3. Let $k : [0,1]^2 \to \mathbb{R}$ be continuous. Let $T : C[0,1] \to C[0,1]$ be given by

$$Tf(s) = \int_0^1 k(s,t)f(t)\,dt \text{ for each } s \text{ in } [0,1].$$

This is known as an *integral operator* and the function k is called the *kernel* of T.

(a) Show that, in fact, $Tf \in C[0,1]$ for each f in $C[0,1]$.

(b) Show that T is linear with $\|T\| \leq \|k\|_\infty$.

(c) Show that any sequence $(f_n)_{n=1}^\infty \subset C[0,1]$ which satisfies

$$\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty \text{ and } \lim_{n\to\infty} \|f_n - Tf_n\|_\infty = 0$$

admits a subsequence for which $f = \lim_{k\to\infty} f_{n_k}$ (uniform limit) is a fixed point of T.