# MATH 351, FALL 2017

**Assignment #6** Due: Dec. 1.

1. Let $(V, \|\cdot\|)$ be a normed vector space.

   **(a)** Show that a finite dimensional subspace $F$ of $V$ is necessarily closed.

   [Hint: show that $F$ is complete.]

   **(b)** Show that a closed subspace $W \subsetneq V$ is nowhere dense in $V$.

   **(c)** Show that if $(V, \|\cdot\|)$ is an infinite dimensional Banach space, then it cannot have a countable basis.

   [Recall that a set $S$ is spanning if any element of $V$ can be realized as a finite linear combination of elements from $S$.]

   **(d)** Deduce that the space $P[0,1]$ of polynomial functions on $[0,1]$ admits no norm with respect to which it is a Banach space.

2. **(a)** Let $(V, \|\cdot\|_V)$ be a Banach space and $(W, \|\cdot\|_W)$ a normed vector space. Show that if a sequence $(T_n)_{n=1}^{\infty} \subset \mathcal{B}(V,W)$ converges pointwise, i.e. $\lim_{n\to\infty} T_n x$ exists in $W$ for each $x$ in $V$, then $\sup_{n\in\mathbb{N}} \|\!|T_n|\!\| < \infty$.

   **(b)** Consider the normed vector space $(\ell_1, \|\cdot\|_2)$. [Recall from A2, $\ell_1 \subset \ell_2$.] Show that the sequence of maps $(\varphi_n)_{n=1}^{\infty} \subset \mathcal{B}(\ell_1, \mathbb{R})$, each given by $\varphi_n(x) = \sum_{k=1}^{n} x_k$, converges pointwise, but that $\sup_{n\in\mathbb{N}} \|\!|\varphi_n|\!\| = \sup_{n\in\mathbb{N}} \sup_{x\in\ell_1, \|x\|_2\leq 1} |\varphi_n(x)| = \infty$.

   [Consider Hölder's inequality with $p = q = 2$ to help compute each $\|\!|\varphi_n|\!\|$.]

3. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces.

   **(a)** Let $f : X \to Y$. Given $\varepsilon > 0$, let

$$D_\varepsilon(f) = \left\{ x \in X : \begin{array}{c} \text{for every } \delta > 0 \text{ there are } y, z \text{ in } B_X(x,\delta) \\ \text{for which } d_Y(f(y), f(z)) \geq \varepsilon \end{array} \right\}.$$

   Show that $D_\varepsilon(f)$ is closed in $X$. Hence deduce that the set of points, $D(f)$, of discontinuity for $f$ is an $\mathrm{F}_\sigma$-set in $X$.

   **(b)** Show that if $(X, d_X)$ is complete, then $D(f)$ is either meager or contains a non-empty open set. Deduce that there is no function $f : \mathbb{R} \to \mathbb{R}$ for which $D(f) = \mathbb{R} \setminus \mathbb{Q}$.

**(c)** Let $f_0, f_n : X \to Y$ $(n \in \mathbb{N})$ be so $\lim_{n\to\infty} f_n(x) = f_0(x)$ for each $x$ in $X$. Then $(f_n)_{n=1}^{\infty}$ is said to *converge uniformly at* $x_0$ if given $\varepsilon > 0$, there are $\delta > 0$ and $N$ in $\mathbb{N}$ for which $d_Y(f_n(x), f_m(x)) < \varepsilon$ whenever $x \in B(x_0, \delta)$.

Show that if $(f_n)_{n=1}^{\infty}$ converges uniformly to $f_0$ at $x_0$, and each $f_n$ is continuous at $x_0$, then $f_0$ is continuous at $x_0$ too.

4. Let $\{q_k\}_{k=1}^{\infty} = (-1,1) \cap \mathbb{Q}$ (i.e. an enumeration of this countable set). Define $f_k, f : (-1,1) \to \mathbb{R}$ by

$$f_k(t) = \begin{cases} (t-q_k)^2 \sin\left(\frac{1}{t-q_k}\right) & \text{if } t \neq q_k \\ 0 & \text{if } t = q_k \end{cases}, \quad f(t) = \sum_{k=1}^{\infty} \frac{1}{2^k} f_k(t).$$

**(a)** Show that $f$ is differentiable on $(-1,1)$.

[Hint: $\sup_{k\in\mathbb{N}} \sup_{t\in(-1,1)} |f_k'(t)| \leq M$ for some $M > 0$. Estimate differences $\left|\frac{f(s)-f(t)}{s-t} - \sum_{k=1}^{\infty} \frac{1}{2^k} f_k'(t)\right|$ for $s$ near $t$. You may liberally use facts about absolutely converging series in $\mathbb{R}$.]

**(b)** Determine the points of discontinuity $D(f')$.

5. A function $f : [0,1] \to \mathbb{R}$ is called *somewhere locally non-decreasing* if there are $a < b$ in $[0,1]$ for which $a \leq x \leq y \leq b$ implies $f(x) \leq f(y)$. There is an analogous definition of *somewhere locally non-increasing.* Then $f$ is called *somewhere locally monotone* if is either locally non-decreasing, or locally non-increasing.

Prove that the family of somewhere locally monotone functions is meager in $(C[0,1], \|\cdot\|_\infty)$.

[Hint: given $\delta \in (0,1]$ consider the set

$$E_\delta^+ = \left\{ f \in C[0,1] : \begin{array}{c} \text{there are } a < b \text{ in } [0,1] \text{ such that } b - a = \delta \\ \text{and } f \text{ is non-decreasing on } [a,b] \end{array} \right\}.]$$

6. **(a)** Let $(X,d)$ be a compact metric space. Show that $(C(X), \|\cdot\|_\infty)$ is separable.

[Hint: if $Z$ is a countable dense subset of $X$, consider $\{d(\cdot, z)\}_{z\in Z} \subset C(X)$.]

**(b)** Is $(C_b(0,1], \|\cdot\|_\infty)$ separable?