MATH 351, FALL 2017

Assignment #5 Due: Nov. 15.

1. **(a)** Suppose $f:(0,1)\to\mathbb{R}$ is differentiable at every point. Show that

$$f \text{ is Lipschitz } \Leftrightarrow \text{ its derivative } f' \text{ is bounded: } \sup_{t\in(0,1)}|f'(t)|<\infty.$$

(b) Classify each $f_k:(0,1)\to\mathbb{R}$ as being *Lipschitz*, *uniformly continuous*, and/or *continuous*:

$$f_1(t)=t\sin\left(\tfrac{1}{t}\right),\qquad f_2(t)=\sin\left(\tfrac{1}{t}\right),\qquad f_3(t)=t^2\sin\left(\tfrac{1}{t}\right).$$

(c) Let $M,\delta>0$ and

$$F_{M,\delta}=\left\{f\in C[0,1]: \begin{array}{c}\text{there is } x \text{ in } [0,1] \text{ so } \frac{|f(t)-f(x)|}{|t-x|}\leq M\\ \text{for } t\in[0,1]\cap[(x-\delta,x)\cup(x,x+\delta)]\end{array}\right\}.$$

(If $f\in F_{M,\delta}$, we say that f is "locally M-Lipschitz" at some x.) Show that $F_{M,\delta}$ is a closed subset of $C[0,1]$ in the topology given by $\|\cdot\|_\infty$.

[Note: given uniformly converging $(f_n)_{n=1}^\infty\subset F_{M,\delta}$, there will be an associated sequence of points of local M-Lipschitzness, $(x_n)_{n=1}^\infty\subset[0,1]$.]

(d) Show that if f is differentiable at some point in $(0,1)$, then $f\in F_{M,\delta}$ for some $M,\delta>0$.

2. Let (Y,d_Y) be a compact metric space.

(a) Show that (Y,d_Y) is separable, i.e. there isa countable set $Z\subseteq Y$ so $\overline{Z}=Y$.

(b) We say that a metric space (X,d_X) is *totally disconnected* if for any $x\neq y$ in X there are open U,V in X so

$$x\in U, y\in V,\quad X=U\cup V\quad \text{and } U\cap V=\varnothing.$$

Show that the Cantor set (C,d) (d is relativized metric from $\mathbb{R}$) is totally disconnected.

[The following notation may be handy. In the notation of the appendix of A3, let $U_{b_1\dots b_n}=C\cap I_{b_1\dots b_n}$.]

(c) (Universality of Cantor set amongst compact metric spaces.) Show that there exists a continuous surjection $f : C \to Y$.

[Hint: find f as the limit, in the metric of A3, Q4, of a sequence of continuous functions, each with finite range.]

(d) Show that if $(Y, d_Y) = ([0,1], d)$ (usual metric), then f, above, cannot be bijective.

(e) (Generalized Peano curves.) Let $(V, \|\cdot\|)$ be a normed vector space. A non-empty $K \subseteq V$ is called *convex* if for any x, y in K, the line segment $\{x + t(y-x) : t \in [0,1]\}$ is a subset of K.

Show that if $K \subset V$ is compact and convex, then there is a continuous surjection $g : [0,1] \to K$. (Hence $g([0,1]) = K$ is a "space-filling curve".)

[Hint: if the elements $f_n : C \to K$, above, are built nicely, then one can build $g_n : [0,1] \to K$ as the piecewise affine extension of f. A function $h : [0,1] \to K$ is *piecewise affine* if there are $0 = t_0 < t_1 < \cdots < t_n = 1$ and $x_0, x_1, \dots, x_n$ (not necessarily distinct) in V so $h(t) = x_{j-1} + \frac{t - t_{j-1}}{t_j - t_{j-1}}(x_j - x_{j-1})$ for $t \in [t_{j-1}, t_j]$.]

3. (Newton's method.)

(a) Let I be closed interval in $\mathbb{R}$, and $f : I \to \mathbb{R}$ satisfy that

$$\begin{aligned} &f \text{ is twice differentiable on some open interval containing } I, \\ &f'(t) \neq 0 \text{ and } |f''(t)f(t)| \leq c|f'(t)|^2 \text{ on } I, \text{ for some } 0 < c < 1. \end{aligned}$$

Show that the if the function $\gamma(t) = t - \dfrac{f(t)}{f'(t)}$ satisfies $\gamma(I) \subseteq I$, then it admits a unique fixed point z, which satisfies $f(z) = 0$.

(b) (Babylonian pth-root estimate.) Let $p \geq 2$ in $\mathbb{N}$ and $a \geq 1$ in $\mathbb{R}$. Verify that the sequence given by $a_0 = a$, then, inductively by

$$a_n = \left(1 - \frac{1}{p}\right) a_{n-1} + \frac{1}{p} \frac{a}{(a_{n-1})^{p-1}}$$

satisfies for each n that

$$0 \leq a_n - \sqrt[p]{a} \leq \left(\frac{p-1}{p}\right)^n \left(a - \frac{1}{a^{p-2}}\right).$$

[Hint: let $f(t) = t^p - a$ and γ be as above on $I = [\sqrt[p]{a}, \infty)$. Show that if $t \in I$, then $\gamma(t) \in I$ too; to get this, let $\alpha = t^{1-1/p}$ and $\beta = (\frac{a}{t^{p-1}})^{1/p}$ and recall the lemma used to prove Hölder's inequlity.]

4. Show that $\Gamma : C[0,1] \to C[0,1]$ given for t in $[0,1]$ by

$$\Gamma(f)(t) = t^2 + \int_0^t s^2 f(s)\, ds$$

is a strict contraction. Starting at $f_0 = 0$, calculate a power series for the fixed point f_{sol} of Γ.

[Hint: the technique is similar to the proof of the Picard-Lindelöf Theorem but with more control: show that $\int_0^t s^2|f(s)|\, ds \leq \frac{1}{3}\|f\|_\infty$.]