MATH 351, FALL 2017

Assignment #4 Due: Oct. 27.

1. (On completions of normed vector spaces) Let $(V, \|\cdot\|_0)$ and $(W, \|\cdot\|_1)$ be normed vector spaces.

   (a) Verify that $\|\!|(v,w)|\!\| = \|v\|_0 + \|w\|_1$ defines a norm on $V \times W$. Moreover, if $(\overline{V}, \bar{d}_0)$ is a completion of (V, d_0) ($d_0(v,x) = \|v - x\|_0$); and $(\overline{V}, \bar{d}_1)$ is a completion of (W, d_1) ($d_1(y,w) = \|y - w\|_1$); then

   $$(\overline{V} \times \overline{W}, \bar{d}) \text{ with } \bar{d}((\bar{x},\bar{y}),(\bar{v},\bar{w})) = \bar{d}_0(\bar{x},\bar{v}) + \bar{d}_1(\bar{y},\bar{w})$$

   is the completion of $(V \times W, d)$ where $d((x,y),(v,w)) = \|\!|(x,y)-(v,w)|\!\|$.

   [Recall that $V \times W$ is a $\mathbb{R}$-vector space with operations $(x,y)+(v,w) = (x+v, y+w)$ and $\alpha(x,y) = (\alpha x, \alpha y)$.]

   (b) Show that $\overline{V}$ is a $\mathbb{R}$-vector space, and the norm $\|\cdot\|_0$ extends to a norm $\|\cdot\|$ on $\overline{V}$.

   [Consider the functions $A : V \times V \to V$, $A(v,w) = v + w$; and $S_\alpha : V \to V$, $S_\alpha(v) = \alpha v$.]

2. (Distance from a set) Let (X,d) be a metric space an $\varnothing \neq A \subseteq X$. Define for x in X, the distance form x to A by

   $$\operatorname{dist}(x,A) = \inf\{d(x,a) : a \in A\}.$$

   (a) Show that $\operatorname{dist}(x,A) = 0$ if and only if $x \in \overline{A}$ (closure).

   (b) Show that $f : X \to \mathbb{R}$, $f(x) = \operatorname{dist}(x,A)$ is continuous.

   (c) Show that there is a sequence $U_1, U_2, \ldots$ of open subsets in (X,d) for which $\overline{A} = \bigcap_{n=1}^{\infty} U_n$.

   (d) Let $\varnothing \neq K \subset X$ be a compact set with $K \cap \overline{A} = \varnothing$. Show that there is a continuous function $g : X \to [0,1]$ such that $g(x) = 0$ for x in A and $g(y) = 1$ for y in K.

Don't forget the next question ...

3. **(a)** Let (X, d_X) and (Y, f_Y) be compact metric spaces. Show that if a map $f : X \to Y$ is both continuous and bijective, then its inverse $f^{-1} : Y \to X$ is also continuous.

   (b) Let C be the Cantor set with relitivized metric from $(\mathbb{R}, |\cdot|)$, and $P = \prod_{k=1}^{\infty}\{0, \frac{1}{2^k}\}$ with relitivized metric from $(\ell_1, \|\cdot\|_1)$. Show that there exists a continuous bijection $f : C \to P$.

   (c) Show that there is a continuous surjection $\varphi : C \to [0, 1]$. [Consider $\varphi = \|f(\cdot)\|_1$.]