MATH 351, FALL 2017

Assignment #3 Due: Oct. 20.

1. Let X be a non-empty set and recall the equivalence relations $\approx$ and $\sim$ on the space of metrics $M(X)$ from the last assignment.

   (a) Give an example of an X and d, ρ in $M(X)$ with $d \not\approx \rho$ but with the property that a sequence $(x_n)_{n=1}^{\infty} \subset X$, is Cauchy in (X, d) if and only if it is Cauchy in (X, ρ).

   (b) Give an example of an X and d, ρ in $M(X)$ with $d \sim \rho$, and there is a sequence $(x_n)_{n=1}^{\infty} \subset X$ which is Cauchy in one of (X, d) or (X, ρ), but not the other.

   [Consider some of the first example of metrics suggested, in lectures.]

2. Let (X, d_X) and (Y, d_Y) be complete metric spaces.

   (a) A subset $F \subseteq X$ is closed in (X, d) $\Leftrightarrow$ the relativized metric space (F, d_F) is complete.

   (b) Let A be a non-empty subset of X. A function $f : A \to Y$ is called *uniformly continuous*† if given $\varepsilon > 0$, there is $\delta > 0$ such that

   $$d_Y(f(x), f(y)) < \varepsilon, \text{ whenever } d_X(x, y) < \delta \text{ for } x, y \text{ in } A.$$

   Show that if $f : A \to Y$ is uniformly continuous, then there is a unique continuous function $\bar{f} : \overline{A} \to Y$ such that $\bar{f}|_A = f$. Moreover, $\bar{f}$ is uniformly continuous.

   † The difference between this definition and that of "continuous" is that this holds uniformly for *all* choices of x, y within small distance of each other, rather than fixing one of x or y in advance.

3. Let (X, d) be a metric space. A subset $P \subseteq X$ is called *perfect* if $P' = P$, i.e. P is the set of accumulation points of itself.

(a) Verify whether or not the following sets below are perfect within the metric spaces suggested. Justify your reasoning.

(i) $[a, b]$ in $(\mathbb{R}, |\cdot|)$ ($a < b$ in $\mathbb{R}$);

(ii) $\{q \in \mathbb{Q} : 0 \leq q \text{ and } q^2 < 2\}$ in $(\mathbb{Q}, |\cdot|)$ [you may take for granted that $\sqrt{2}$ is irrational];

(iii) the Cantor set C in $(\mathbb{R}, |\cdot|)$ [see Appendix to this assignment];

(iv) $\prod_{k=1}^{\infty}\{0, \frac{1}{2^k}\}$ in $(\ell_1, \|\cdot\|_1)$.

(b) Let P be a non-empty perfect set in a metric space (X, d). Verify that there are two points x_0, x_1 in P and an $r_1 > 0$ for which $B[x_0, r_1] \cap B[x_1, r_1] = \varnothing$. Furthermore, for each b_1 in $\{0, 1\}$ there are points $x_{b_1 0}$ and $x_{b_1 1}$ in the punctured ball $B(x_{b_1}, r_1) \setminus \{x_{b_1}\}$ and $r_2 > 0$ for which

$$\begin{aligned} &B[x_{b_1 b_2}, r_2] \subset B(x_{b_1}, r_1) \setminus \{x_{b_1}\} \text{ for } b_2 \in \{0, 1\} \\ &\text{and } B[x_{b_1 0}, r_2] \cap B[x_{b_1 1}, r_2] = \varnothing. \end{aligned}$$

(c) Show that a non-empty perfect set P in a complete metric space (X, d) has $|P| \geq \mathfrak{c}$.

[Hint: iterate the procedure of (b). Consider, for each d in $\{0, 1\}^{\mathbb{N}}$, the sequence of points $(x_{b_1}, x_{b_1 b_2}, x_{b_1 b_2 b_3}, \dots) \subset P$ thus created.]

(d) Does the conclusion of (c), above, hold if we do not assume that (X, d) is complete?

4. Let (X, d_X) and (Y, d_Y) be metric spaces. Given non-empty $B \subseteq Y$ we let $\operatorname{diam}_Y(B) = \sup_{y_1, y_2 \in B} d_Y(y_1, y_2)$. Let

$$C_b(X, Y) = \{f \in Y^X : f \text{ is continuous and } \operatorname{diam}_Y\big(f(X)\big) < \infty\}$$

where $f(X) = \{f(x)' x \in X\}$ is the range of f.

(a) Show that $d_C : C_b(X, Y) \times C_b(X, Y) \to [0, \infty)$ given by

$$d_C(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$$

defines a metric on $C_b(X, Y)$. [Is it obvious that $d_C(f, g) < \infty$?]

(b) Show that if (Y, d_Y) is complete, then $(C_b(X, Y), d_C)$ is also complete.

Appendix: construction of the Cantor set. To establish consistent notation, let us recall construction of the Cantor set in $\mathbb{R}$.

Let $I = [0, 1]$ and $J = (\frac{1}{3}, \frac{2}{3})$, and $C_1 = I \setminus J = I_0 \cup I_1$, where $I_0 = [0, \frac{1}{3}]$ and $I_1 = [\frac{2}{3}, 1]$.

We continue inductively.

Having constructed $C_n = \bigcup_{(b_1,\dots,b_n)\in\{0,1\}^n} I_{b_1\dots b_n}$, where each $I_{b_1\dots b_n}$ is a closed interval of length $\frac{1}{3^n}$, let $J_{b_1\dots b_n}$ be the open middle third of $I_{b_1\dots b_n}$ and let

$$C_{n+1} = C_n \setminus \bigcup_{(b_1,\dots,b_n)\in\{0,1\}^n} J_{b_1\dots b_n} = \bigcup_{(b_1,\dots,b_n,b_{n+1})\in\{0,1\}^{n+1}} I_{b_1\dots b_n b_{n+1}}.$$

Here each $I_{b_1\dots b_n 0}$ and $I_{b_1\dots b_n 1}$ are the left and right closed thirds of $I_{b_1\dots b_n}$, each of length $\frac{1}{3^{n+1}}$.

We let $C = \bigcap_{n=1}^{\infty} C_n$, which is evidently closed. It follows the Nested Intervals Theorem that $C \neq \varnothing$. Indeed, $\varnothing \neq I_0 \cap I_{00} \cap I_{000} \cap \dots \subset C$.

We may see much more concretely that $C \neq \varnothing$. Indeed if $(t_k)_{k=1}^{\infty} \in \{0, 2\}^{\mathbb{N}}$ then

$$\sum_{k=1}^{\infty} \frac{t_k}{3^k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{t_k}{3^k} \in \bigcap_{n=1}^{\infty} I_{\frac{t_1}{2}\dots\frac{t_n}{2}}.$$

Indeed, each $\sum_{k=1}^{n} \frac{t_k}{3^k}$ is the left endpoint of $I_{\frac{t_1}{2}\dots\frac{t_n}{2}}$.