MATH 351, FALL 2017

Assignment #2 Due: Oct. 6.

1. (On equivalence of norms) On a $\mathbb{R}$-vector space, an $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ be two norms defined on V. We let

$$\|\cdot\| \approx |\!|\!|\cdot|\!|\!| \quad \Leftrightarrow \quad \begin{array}{l}\text{there are } m, M > 0 \text{ for which} \\ m\|x\| \leq |\!|\!|x|\!|\!| \leq M\|x\| \text{ for all } x \text{ in } V.\end{array}$$

(a) Verify that $\approx$ is an equivalence relation on the set $N(V)$ of norms on V.

(b) Consider the vector space $\mathbb{R}^n$. Show that for $1 < p < r < \infty$ that

$$\|x\|_\infty \leq \|x\|_r \leq \|x\|_p \leq \|x\|_1 \leq n\|x\|_\infty \text{ for } x \text{ in } \mathbb{R}^n$$

and deduce that these norms are all equivalent.

[Hint: for the hardest of these inequlities, if $x \neq 0$, first divide $\|x\|_r{}^r$ by $\|x\|_\infty{}^r$, notice that $t^r \leq t^p$ if $0 \leq t \leq 1$.]

(c) Show that for x in $\mathbb{R}^n$ we have $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$.

(d) Show that for $1 \leq p < r < \infty$ we have a proper containment relations

$$\ell_p \subsetneq \ell_r \subsetneq c_0.$$

[The integral test for series convergnece is your friend.]

(e) Show that for $1 \leq p < r \leq \infty$, $\|\cdot\|_p \not\approx \|\cdot\|_r$ on ℓ_1. (Recall, from above, that $\ell_1 \subset \ell_p$ for each $1 \leq p \leq \infty$.)

2. (On equivalence of metrics)

Let X be a non-empty set and $M(X) \subset [0,\infty)^{X\times X}$ denote the set of all metrics on X. For d, ρ in $M(X)$ let

$$d \approx \rho \quad \Leftrightarrow \quad \begin{cases}\text{there are } m, M > 0 \text{ for which} \\ md(x,y) \leq \rho(x,y) \leq Md(x,y) \text{ for all } x, y \text{ in } X\text{; and.}\end{cases}$$

$$d \sim \rho \quad \Leftrightarrow \quad \begin{cases}\lim_{n\to\infty} x_n = x \text{ in } (X,d) \;\Leftrightarrow\; \lim_{n\to\infty} x_n = x \text{ in } (X,\rho) \\ \text{for any sequence } (x_n)_{n=1}^\infty \subseteq X \text{ and any } x \text{ in } X\end{cases}$$

(a) Verify that $\approx$ and $\sim$ are equivalence relations on $M(X)$.

(b) Show that $d \approx \rho$ in $M(X)$ implies that $d \sim \rho$

(c) Show that $d \sim \rho$ in $M(X) \Leftrightarrow (X, d)$ and (X, ρ) admit the same open sets.

(d) Let $d \in M(X)$ and $f : [0, \infty) \to [0, \infty)$ satisfy that

$f(0) = 0$ and f is strictly increasing, subadditive and continuous. $(\heartsuit)$

Show that $d_f : X \times X \to [0, \infty)$, $d_f(x, y) = f(d(x, y))$ defines a metic with $d_f \sim d$. Furthermore, show that $f(s) = \frac{s}{s+1}$ satisfies $(\heartsuit)$.

(e) Does $d \sim \rho$ in $M(X)$ imply that $d \approx \rho$? Prove this, or supply a counterexample.

3. We say that a metric space (X, d) is *separable* if there is a countable set $Z = \{z_k\}_{k=1}^{\infty} \subseteq X$ with closure $\overline{Z} = X$.

   (a) Show that if (X, d) is separable, then it cardinality satisfies $|X| \leq \mathfrak{c}$.

   [Hint: this has aspects similar to our construction of $\mathbb{R}$ from $\mathbb{Q}$.]

   (b) Show that for $1 \leq p < \infty$ that ℓ_p is separable.

   (c) Let $C(\mathbb{R}) = \{f \in \mathbb{R}^{\mathbb{R}} : f \text{ is continuous}\}$. Show that $|C(\mathbb{R})| = \mathfrak{c}$.

   [Hint: a continuous function is determined by its behaviour on $\mathbb{Q}$.]

   (d) Show that ℓ_∞ is not separable.

   [Hint: find a subset of X elements which is uncountbale and $\|\chi - \chi'\|_\infty \geq 1$ for $\chi \neq \chi'$ in X.]

   (e) Show that $|\ell_\infty| = \mathfrak{c}$.

   [Hint: find an injection $\ell_\infty \hookrightarrow C(\mathbb{R})$.]