MATH 351, FALL 2017

Assignment #1 Due: Sept. 22.

1. Consider the space of Cauchy sequences of rationals:

$$X = \left\{ (q_n) = (q_n)_{n=1}^{\infty} \in \mathbb{Q}^{\mathbb{N}} : \begin{array}{l} \text{for each } \varepsilon \text{ in } \mathbb{Q}_+, \text{ there is } n_\varepsilon \text{ in } \mathbb{N} \text{ such} \\ \text{that } |q_m - q_n| < \varepsilon \text{ whenever } m, n \geq n_\varepsilon \end{array} \right\}.$$

In X we let

$$\begin{array}{l} (q_n) \sim (r_n) \text{ if for any } \varepsilon \text{ in } \mathbb{Q}_+, \text{ there is } n_\varepsilon \text{ in } \mathbb{N} \text{ for which} \\ \qquad\qquad |q_n - r_n| < \varepsilon \text{ whenever } n \geq n_\varepsilon; \\ (q_n) \leq (r_n) \text{ if for any } \varepsilon \text{ in } \mathbb{Q}_+, \text{ there is } n_\varepsilon \text{ in } \mathbb{N} \text{ for which} \\ \qquad\qquad q_n \leq r_n + \varepsilon, \text{ in } \mathbb{Q}, \text{ whenever } n \geq n_\varepsilon; \text{ and} \\ (q_n) < (r_n) \text{ if } (q_n) \leq (r_n) \text{ and } (q_n) \not\sim (r_n). \end{array}$$

(a) Suppose $(q_n) < (r_n)$ in X. Show that there is q in $\mathbb{Q}$ such that the constant sequence $(q) = (q, q, \dots)$ satisfies $(q_n) < (q) < (r_n)$.

(b) Given (q_n) in X, show that there is q in $\mathbb{Q}$ such that $(q_n) < (q)$.

2. Let A be a non-empty set.

(a) Show that $\mathbb{N} \preceq A \Leftrightarrow A$ is partitionable into denumerable sets: i.e. there is a family $\{P_i\}_{i \in I} \subset \mathcal{P}(A)$ such that

- each P_i is denumerable,
- $P_i \cap P_j = \varnothing$ if $i \neq j$ in I, and
- $A = \bigcup_{i \in I} P_i$.

[Hint: the non-trivial direction requires Z.L.]

(b) Show that the following are equivalent:

(i) $\aleph_0 \leq |A|$, i.e. A is infinite;

(ii) $|A| = n|A|$ for any $n \geq 2$ in $\mathbb{N}$; and

(iii) $|A| = \aleph_0 |A|$.

Don't forget next page.

(c) Deduce that a pair of sets B, C, with at least one infinite, satisfies $|B| + |C| = \max\{|B|, |C|\}$.

Note that $\max\{|B|, |C|\}$ makes sense, thanks to the Comparison Lemma.

(d) Let B and C be non-empty sets. A map $\varphi : B \to C$ is *countable-to-one* if for every c in C, $\varphi^{-1}(\{c\})$ is countable. Show that if such a map exists then $|B| \leq \aleph_0 |C|$.

3. Let A be an infinite set.

   (a) Show that $|A| = |A|^n$ for any $n \geq 2$ in $\mathbb{N}$.

   [Hint: Consider $n = 2$, first. Adapt proof of comparison lemma, generally.]

   (b) Deduce that a pair of non-empty sets B, C, with at least one infinite, satisfies $|B||C| = \max\{|B|, |C|\}$.

   (c) Prove that the set $\mathcal{F}(A)$ of finite subsets of A satisfies $|\mathcal{F}(A)| = |A|$.

   (d) Let V be a vector space over a field $\mathbb{K}$ which admits an infinite linearly independent set. Show that any two bases B and B' have the same cardinality.

   [Hint: Apply q. 2 (d), above to $\mathcal{F}(B)$ and $\mathcal{F}(B')$.]

   This cardinal above is called the *dimension* of V over $\mathbb{K}$, denoted $\dim_{\mathbb{K}} V$.

   (e) Without appealing to the continuum hypothesis, determine $\dim_{\mathbb{Q}} \mathbb{R}$.