Problem set on Implicit Function Theorem and Lagrange Multipliers

1. Verify the following corollaries of the Implicit Function Theorem:

   (a) If $\varnothing \neq U \subset \mathbb{R}^2$ is open, $f \in \mathcal{C}^1(U, \mathbb{R})$, $(x_0, y_0) \in U$ so $f(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$, then there is a neigbourhood $V \subset \mathbb{R}$ of x_0 and a function $\varphi \in \mathcal{C}^1(V, \mathbb{R})$ such that for $x \in V$

   $$f(x, \varphi(x)) = 0, \quad \varphi(x_0) = y_0 \quad \text{and} \quad \varphi'(x) = -\frac{\frac{\partial f}{\partial x}(x, \varphi(x))}{\frac{\partial f}{\partial y}(x, \varphi(x))}.$$

   (b) If $\varnothing \neq U \subset \mathbb{R}^3$ is open, $f \in \mathcal{C}^1(U, \mathbb{R})$, $(x_0, y_0, z_0) \in U$ so $f(x_0, y_0, z_0) = 0$ and $\frac{\partial f}{\partial z}(x_0, y_0, z_0) \neq 0$, then there is a neigbourhood $V \subset \mathbb{R}^2$ of (x_0, y_0) and a function $\varphi \in \mathcal{C}^1(V, \mathbb{R})$ such that for $(x, y) \in V$

   $$f(x, y, \varphi(x, y)) = 0, \quad \varphi(x_0, y_0) = z_0,$$
   $$\frac{\partial \varphi}{\partial x}(x, y) = -\frac{\frac{\partial f}{\partial x}(x, y, \varphi(x, y))}{\frac{\partial f}{\partial z}(x, y, \varphi(x, y))} \quad \text{and} \quad \frac{\partial \varphi}{\partial y}(x, y) = -\frac{\frac{\partial f}{\partial y}(x, y, \varphi(x, y))}{\frac{\partial f}{\partial z}(x, y, \varphi(x, y))}.$$

   (c) In each of the situations above, verify the the expressions for the partial derivatives by applying the Chain Rule directly to the expressions $f(x, \varphi(x)) = 0$ and $f(x, y, \varphi(x, y)) = 0$.

2. **(a)** Show that the equations

   $$x^2 - yu = 0 \qquad xy + uv = 0 \qquad (*)$$

   which admit solution $(x_0, y_0, u_0, v_0) = (2, 4, 1, -8)$, define implicit functions $u = u(x, y)$, $v = v(x, y)$ in a neighbourhood of $(x_0, y_0) = (2, 4)$ which satisfy the equations $(*)$ and the system of *partial differential equations*

   $$\frac{\partial u}{\partial x} = \frac{2x}{y}, \ \frac{\partial u}{\partial y} = -\frac{u}{y}, \ \frac{\partial v}{\partial x} = -\frac{y}{u} - \frac{2vx}{uy} \ \text{ and } \ \frac{\partial v}{\partial y} = -\frac{x}{u} + \frac{v}{y}.$$

   (b) Use the equations $\frac{\partial u}{\partial x} = \frac{2x}{y}$ and $xy + uv = 0$ in a neighbourhood of the point $(x_0, y_0, u_0, v_0) = (2, 4, 1, -8)$ to find explicit formulas for $u(x, y)$ and $v(x, y)$. Indicate the domain of $\varphi(x, y) = \big(u(x, y), v(x, y)\big)$.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

   $$f(x, y, z) = (x^3 + yz, xz + y^3).$$

   Show that at any point (x_0, y_0, z_0) such that $x_0 = -y_0, z_0 = x_0^2$ and $x_0 \neq 0$, there are $\mathcal{C}^1$ functions φ_1, φ_2 defined on a neighbourhood of x_0 so that $\varphi_1(x_0) = y_0$, $\varphi_2(x_0) = z_0$ and $f(x, \varphi_1(x), \varphi_2(x)) = 0$. Determine $J_\varphi(x, \varphi(x))$, where $\varphi(x) = \big(\varphi_1(x), \varphi_2(x)\big)$.

Lots of Lagrange Multipliers

4. Let $S = P \cap C$ where

$$P = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\} \text{ (plane)}$$
$$C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\} \text{ (cylinder)}.$$

   Find the point(s) on S closest to the origin $(0, 0, 0)$.

   [Hint: Paramaterise the plane $z = -x-y$ and minimise $f(x, y) = \|(x, y, -x - y)\|^2$ subject to the appropriate constraint.]

5. Find the maximim and minimum values of $2x^3 + y^3 + 2x$ on $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

   [Hint: Treat 2 types of cases: interior points and boundary points.]

6. Show that the rectangular box of maximal volume, whose sides are parallel to the axes and which may be inscribed on the ellipsoid

$$E = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

   $(a, b, c > 0)$, has corners $\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{b}{\sqrt{3}}, \pm\frac{c}{\sqrt{3}}\right)$.

   [Hint: We've seen this before. Use Lagrange Multipliers this time.]

7. **(a)** Show that if $f, g \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ are such that the curves

$$C_f = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\} \quad \text{and} \quad C_g = \{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\}$$

   do not intersect, and if there the function $d : C_f \times C_g \to \mathbb{R}$ given by $d(x, y, u, v) = \|(x, y) - (u, v)\|$ has an extremeum (maximum or minimum) (x_0, y_0, u_0, v_0), then this point satisfies

$$\frac{x_0 - u_0}{y_0 - v_0} = \frac{\frac{\partial f}{\partial x}(x_0, y_0)}{\frac{\partial f}{\partial y}(x_0, y_0)} = \frac{\frac{\partial g}{\partial x}(u_0, v_0)}{\frac{\partial g}{\partial y}(u_0, v_0)}.$$

   [Hint: If don't square d, you'll get a headache.]

   (b) Use (a) to find the shortest and distance between points of

   the ellipse $E = \{(x, y) \in \mathbb{R}^2 : x^2 + 2xy + 5y^2 - 16y = 0\}$ and
   the line $L = \{(x, y) \in \mathbb{R}^2 : x + y = 8\}$.