MATH 247, WINTER 2007

Assignment #6 Due: April 2

1. Compute the *Taylor Polynomial* of degree 2 for $f(x,y) = \cos(x+y^2)$ about $(0,0)$ as well as the *remainder term.*

2. Let for (x,y) in $\mathbb{R}^2$

$$f(x,y) = x^4 + y^4 \quad \text{and} \quad g(x,y) = x^4 - y^4$$

(a) Show that at f's stationary point $(0,0)$, $\text{Hess}f(0,0)$ is *not* positive definite, but that $(0,0)$ is still a local minimum for f. [Hint: Don't work too hard on this one.]

(b) Show that at g's stationary point $(0,0)$, $\text{Hess}g(0,0)$ is *not* indefinite, but that $(0,0)$ is still a saddle point for g.

3. Let

$$f : \{(x,y) \in \mathbb{R}^2 : x \neq 0 \neq y\} \to \mathbb{R}, \quad f(x,y) = \frac{1}{y} - \frac{1}{x} - 4x + y.$$

Find and classify each of the stationary points of f.

4. Let $\varnothing \neq U \subseteq \mathbb{R}^N$ be open and convex, and $f : U \to \mathbb{R}$ be a continuously twice differentiable function such that

(i) f has one stationary point in $x_0 \in U$, and

(ii) $\text{Hess}f(x)$ is positive definite for all $x \in U$.

Show that $f(x) > f(x_0)$ for all $x \in U \setminus \{x_0\}$, i.e. x_0 is the unique *global minimum* for f in U.

Don't forget the other side ...

5. *Linear Regression*

   Suppose we are given a set of data points $\{(x_1, y_1), \ldots, (x_p, y_p)\}$ – possibly as the result of an experiment – which we expect to have a linear relationship. Thus we would like to find the line $y = ax + b$ which most closely fits this data set. One measure of closeness is that of "least squares": we choose a and b such that

$$\sum_{i=1}^{p}(ax_i + b - y_i)^2$$

   is as small as possible.

   Show that the choice of (a, b) which minimises the least squares measure satisfies the pair of linear equations

$$\begin{aligned}\left(\sum_{i=1}^{p} x_i^2\right)a + \left(\sum_{i=1}^{p} x_i\right)b &= \sum_{i=1}^{p} x_i y_i\\ \left(\sum_{i=1}^{p} x_i\right)a + \qquad\qquad pb &= \sum_{i=1}^{p} y_i.\end{aligned}$$

   Show, moreover, that this pair of linear equations always admits a solution if the numbers $x_1, \ldots, x_p$ are distinct.

   [Hint: You may need the inequality $(\sum_{i=1}^{p} x_i)^2 \leq p\sum_{i=1}^{p} x_i^2$, which can be proved using Cauchy-Schwarz. When is this inequality strict?]

   Note: Similar technique may be used to find quadratic, cubic or higher order approximating curves.

6. Show that the rectangular box of maximal volume, whose sides are parallel to the axes and which may be inscribed on the ellipsoid

$$E = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

   $(a, b, c > 0)$, has corners $\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{b}{\sqrt{3}}, \pm\frac{c}{\sqrt{3}}\right)$.

   [Hint: This is really a 2-variable problem over a compact region.]