MATH 247, FALL 2005

Assignment #6 Due: March 21.

1. **(a)** Show that if a bounded set $S \subset \mathbb{R}^N$ has only finitely many cluster points, then S has *content zero*.

   (b) Let $C_0 = [0,1]$, $C_1 = C_0 \setminus (\frac{1}{3}, \frac{2}{3})$, $C_2 = C_1 \setminus \left[(\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})\right]$, etc. Hence the set C_n is comprised of 2^n compact intervals, and C_{n+1} is created by removing the open middle third of each component interval of C_n. Let
   $$C = \bigcap_{n=0}^{\infty} C_n.$$
   Show that C has content zero.

   (c) Show that that C is uncountable. [Hint: write every element $t \in [0,1]$ in *ternary* form — i.e. $t = \sum_{k=1}^{\infty} \frac{\tau_k}{3^k}$ where $\tau_k = 0, 1, 2$ — and try to determine which of those are in C.]

   (d) Show that $C' = C$, where C' is the set of cluster points.

2. Let $I \subset \mathbb{R}^N$ be a non-degenerate compact interval, $f : I \to \mathbb{R}$ be a function.

   (a) Show that if f is Riemann integrable on I, then f must be bounded. [Hint: Cauchy with fixed partition.]

   (b) Show that if f is Riemann integrable on I, then f^2 (where $f^2(x) = f(x)^2$) is also Riemann integrable on I. [Hint: Cauchy, again.]

   (c) Deduce that if $f, g : I \to \mathbb{R}$ are Riemann integrable, then so too is fg (where $fg(x) = f(x)g(x)$). [Hint: show $(f-g)^2$, $(f+g)^2$ are integrable, first.]

3. **(a)** Let $I \subset \mathbb{R}^N$ be a compact interval, $f : I \to \mathbb{R}$ be a continuous function. If $\mathcal{P}$ is a partition of I, with family of non-overlapping intervals $\{I_\nu\}_{\nu \in N(\mathcal{P})}$, let
   $$\ell(\mathcal{P}) = \max\big\{\ell(I_\nu) : \nu \in N(\mathcal{P})\big\}$$
   where $\ell\big([a_1, b_1] \times \cdots \times [a_N, b_N]\big) = \max\{b_j - a_j : j = 1, \ldots, N\}$.

   Show that given $\varepsilon > 0$, there is $\delta > 0$ such that if $\mathcal{P}$ is a partition of I with $\ell(\mathcal{P}) < \delta$, then for any Riemann sum $S(f, \mathcal{P})$ we have $\left|S(f,\mathcal{P}) - \int_I f\right| < \varepsilon$. Hence we can write "$\int_I f = \lim_{\ell(\mathcal{P}) \to 0} S(f, \mathcal{P})$".

(b) Provide a counter-example to show that $\lim_{\mu(\mathcal{P})\to 0} S(f,\mathcal{P})$ need not exist for a continuous function f on a compact interval I. Here $\mu(\mathcal{P}) = \max\{\mu(I_\nu) : \nu \in N(\mathcal{P})\}$.

4. Evaluate the iterated integrals:

   (a) $\displaystyle\int_0^\pi \int_x^\pi \frac{\sin y}{y} dydx$

   (b) $\displaystyle\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} e^x \, dxdy$

   (c) $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sin\left(\frac{\pi y}{\sqrt{1-x^2}}\right) dxdy$

   NOTE. A bounded function need not be defined on an entire domain to be Riemann integrable there; it only needs to be defined on the complement of a set of content 0.

5. Let $D = [0,\infty)\times[0,2\pi)\times[0,\pi]$ and $p : D \to \mathbb{R}^3$ be the *spherical coordinatisation* given by

   $$p(r,\theta,\phi) = \big(x(r,\theta,\phi), y(r,\theta,\phi), z(r,\theta,\phi)\big) := (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi).$$

   (a) Compute $\det J_p(r,\theta,\phi)$ for $(r,\theta,\phi) \in D^\circ$.

   (b) Consider the spherical and flat slices of a cone

   $$C_1 = \big\{(x,y,z) \in \mathbb{R}^3 : \sqrt{x^2+y^2} \le z \text{ and } 1 \le \|(x,y,z)\| \le 2\big\} \text{ and}$$
   $$C_2 = \big\{(x,y,z) \in \mathbb{R}^3 : \max\{1, \sqrt{x^2+y^2}\} \le z \le 2\big\}.$$

   Compute

   $$\int_{C_1} \frac{1}{x^2+y^2+z^2} d\mu(x,y,z) \quad \text{and} \quad \int_{C_2} \frac{1}{x^2+y^2+z^2} d\mu(x,y,z).$$

   [Some spherical equations: spheres centred at the origin, $r = r_0$; cones centred on the z-axis, $\phi = \phi_0$, $0 < \phi_0 < \pi/2$; planes parallel to the xy-plane, $z_0 = r\cos\phi$ where $z_0 \ne 0$.]

   NOTE. The integral $\int_C \frac{Gm\rho}{\|(x,y,z)\|^2} d\mu(x,y,z)$ computes the force due to gravity, on a point mass m at the origin, from an object C of uniform density ρ.