MATH 247, WINTER 2007

Assignment #4 Due: March 2

1. Show that each of the following $f : [0, \infty) \to \mathbb{R}$ are uniformly continuous.

   (a) $f(x) = \sqrt{x}$ [Hint: $[0, \infty) = [0, 1] \cup [1 - \varepsilon, \infty)$.]

   (b) $f(x) = \dfrac{\sin(x^4)}{1+x}$ [Hint: $[0, \infty) = [0, N + 1] \cup (N, \infty)$ for large N.]

2. Let

$$f : \mathbb{R}^2 \to \mathbb{R} \text{ be given by } f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{otherwise.} \end{cases}$$

   Show that f is twice partially differentiable, but that

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) \neq \frac{\partial^2 f}{\partial y \partial x}(0, 0).$$

   Does this contradict the Mixed Partials Theorem (Theorem 3.2.5, in the lecture notes)?

3. Let $f, g : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \text{ and } g(x, y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

   (a) Show that the *directional derivative*, $D_v f(0, 0)$ for $v \in \mathbb{R}^2$ with $\|v\| = 1$, exists if and only if $v = e_1 = (1, 0)$ or $v = e_2 = (0, 1)$.

   (b) Show that the directional derivative $D_v g(0, 0)$ exists for every $v \in \mathbb{R}^2$ with $\|v\| = 1$, but that

$$D_v g(0, 0) \neq \nabla g(0, 0) \cdot v$$

   in general. Hence deduce that g is not differentiable.

4. *Product rules*

(a) Let $\varnothing \neq D \subseteq \mathbb{R}^N$, $x_0 \in D^\circ$ and $f, g : D \to \mathbb{R}^M$ each be differentiable at x_0. Let $\varphi = f \cdot g : D \to \mathbb{R}$, i.e. $\varphi(x) = f(x) \cdot g(x)$ (dot product). Show that φ is differentiable at x_0 and that for any v in $\mathbb{R}^N$

$$D\varphi(x_0)(v) = f(x_0) \cdot Dg(x_0)(v) + Df(x_0)(v) \cdot g(x_0).$$

[You may need to employ a "divide and conquor" strategy, as in the proof of the Chain Rule.]

(b) Formulate and prove a *product rule* in the following situation: $\varnothing \neq D \subseteq \mathbb{R}^N$, $x_0 \in D^\circ$, $f : D \to \mathbb{R}^M$, $\varphi : D \to \mathbb{R}$ and $g = \varphi f : D \to \mathbb{R}^M$, i.e. $g(x) = \varphi(x) f(x)$ (scalar multiplication).

5. *Chain rule*

(a) Let $\varnothing \neq D \subseteq \mathbb{R}^N$, $p = (p_1, \dots, p_M) : D \to \mathbb{R}^M$ be differentaible on $U = D^\circ$, and $f : \mathbb{R}^M \to \mathbb{R}$ be differentiable. Show that $g = f \circ p : D \to \mathbb{R}$ is differentiable, and for any x_0 in U we have

$$\frac{\partial g}{\partial x_j}(x_0) = \sum_{i=1}^{M} \frac{\partial f}{\partial p_i}\big(p(x_0)\big) \frac{\partial p_i}{\partial x_j}(x_0)$$

for $j = 1, \dots, N$. We often write "$\frac{\partial g}{\partial x_j} = \sum_{i=1}^{M} \frac{\partial f}{\partial p_i} \circ p \, \frac{\partial p_i}{\partial x_j}$ on U".

(b) Let $p : [0, \infty) \times [0, 2\pi) \to \mathbb{R}^2$ be the *polar coordinatisation* given by $p(r, \theta) = \big(x(r, \theta), y(r, \theta)\big) = (r \cos\theta, r \sin\theta)$. If $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, and $g = f \circ p$, show that (with a slight abuse of notation)

$$\frac{\partial g}{\partial r} = \frac{1}{\sqrt{x^2 + y^2}} \left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) \quad \text{and} \quad \frac{\partial g}{\partial \theta} = x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x}.$$

(c) Let $I \subset \mathbb{R}$ be an open interval, and $\gamma = (\gamma_1, \dots, \gamma_N) : I \to \mathbb{R}^N$ be a differentiable function, i.e. so the derivatives γ_j' exist, and set $\gamma' = (\gamma_1', \dots, \gamma_N')$. If $f : \mathbb{R}^N \to \mathbb{R}$ is differentiable in a neighbourhood of γ, show that $g = f \circ \gamma : I \to \mathbb{R}$ is differentiable and

$$g'(t) = \nabla f\big(\gamma(t)\big) \cdot \gamma'(t) \quad \text{for each } t \text{ in } I.$$

(d) Deduce the following result of L. Euler: Suppose $f : \mathbb{R}^N \to \mathbb{R}$ is *homogeneous of degree* m, i.e. for any t in $\mathbb{R}$, x in $\mathbb{R}^N$ we have $f(tx) = t^m f(x)$. Then for $x = (x_1, \ldots, x_N)$ in $\mathbb{R}^N$

$$\sum_{j=1}^{N} x_j \frac{\partial f}{\partial x_j}(x) = m f(x)$$

Note: the functions given by $f(x,y,z) = xyz + x^2 z$ $(m = 3)$; $f(x,y) = \frac{x^2 y}{x^2+y^2}$, $(x,y) \neq 0$ $(m = 1)$ are each homogeneous of respective degree m.

Bonus Question. Prove that the set in $\mathbb{R}^2$ given by

$$S = \{(0,y) : -1 \leq y \leq 1\} \cup \{(x, \sin(1/x)) : 0 < x \leq 1\}$$

is closed and connected, but not path connected.