PMATH 247, WINTER 2007

Assignment #3 Due: February 9

1. If S, T are non-empty subsets of $\mathbb{R}^N$ then we define

$$\text{dist}(S,T) := \inf\{\|x-y\| : x \in S \text{ and } y \in T\}.$$

(a) Find examples of *closed* sets $F, G \subset \mathbb{R}$ ($N = 1!$) for which $\text{dist}(F,G) = 0$ but $F \cap G = \varnothing$.

(b) Show that if closed sets $F, G \subset \mathbb{R}$ are *connected* and $F \cap G = \varnothing$ then $\text{dist}(F,G) > 0$.

(c) Find examples of *bounded* sets $K, L \subset \mathbb{R}^N$ for which $\text{dist}(K,L) = 0$ but $K \cap L = \varnothing$. Can one of these be closed?

2. If $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^N$ which converges to x_0 in $\mathbb{R}^N$, show that $K = \{x_n\}_{n=0}^{\infty}$ is compact.

3. **(a)** If $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}^N$ for which there is a number $\theta < 1$ such that

$$\|x_{n+1} - x_n\| \leq \theta \|x_n - x_{n-1}\| \text{ for each } n \geq 2 \qquad (\spadesuit)$$

then $(x_n)_{n=1}^{\infty}$ converges.

(b) Can we still conclude the result of (a) if ($\spadesuit$) is replaced by

$$\|x_{n+1} - x_n\| < \|x_n - x_{n-1}\| \text{ for each } n \geq 2?$$

4. Let $U \subseteq \mathbb{R}^N$ be open. Show that for $f : U \to \mathbb{R}$,

f is continuous on U $\Leftrightarrow$ for each λ in $\mathbb{R}$ both of the sets $\{x \in U : f(x) > \lambda\}$ and $\{x \in U : f(x) < \lambda\}$ are open.

[Hint: inverse images have nice properties with respect to unions and intersections.]

5. Show that if $f(x,y) = xy + x^4 - y^4$, then the equation $f(x,y) = 0$ has at least four solutions on the circle $C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = R^2\}$, where $R > 0$.

Don't forget the next question ...

6. **(a)** Show that if C is a closed subset of $\mathbb{R}^N$ which satisfies the condition that

$$x, y \in C \quad \Rightarrow \quad \frac{1}{2}(x+y) \in C$$

then C is convex.

[Hint: if $\lambda \in [0,1]$, then it has a *binary* representation $\lambda = 0.\varepsilon_1\varepsilon_2\ldots$.]

(b) Can the conclusion of (a) hold if C is not closed? Prove or provide counter-example.

Bonus Question. The set $\left\{(\sin n, \cos\sqrt{n})\right\}_{n=1}^{\infty}$ in $\mathbb{R}^2$ has at least one cluster point. (Why?) Find one.