# PMATH 247, WINTER 2007

**Assignment #2** Due: January 31

1. Determine if the following sets are *open*, *closed* or *neither*. Justify your answers.

   **(a)** $(a_1, b_1)\times\ldots\times(a_N, b_N) \subset \mathbb{R}^N$ where $a_j < b_j$ in $\mathbb{R}$ for $j = 1, \ldots, N$.

   **(b)** $\{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 + \cdots + x_N \in \mathbb{Q}\}$

   **(c)** $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$.

2. Let $S \subseteq \mathbb{R}^N$. Let the *derived set* be given by

   $$S' = \{x \in \mathbb{R}^N : x \text{ is a cluster point of } S\}.$$

   **(a)** Show that if $x_0 \in S'$, then every neighbourhood $U$ of $x_0$ contains an infinite number of points from $S$.

   **(b)** Show that $S'$ is closed.

   **(c)** A set $F$ is called *perfect* if $F = F'$. Find examples of closed sets, one which is perfect, and one which is not.

3. For each of the following sets $S$, state (without proof) $\overline{S}$, $S^\circ$, $S'$, $\partial S$, i.e. the *closure*, *interior*, *derived set* and *boundary*, respectively

   **(a)** $S_1 = \{x \in \mathbb{R}^N : 0 < \|x\| \leq 1\}$

   **(b)** $S_2 = \left\{\left(\frac{1}{m}, \frac{1}{n}\right) : m, n \in \mathbb{N}\right\} \subset \mathbb{R}^2$

   **(c)** $S_3 = \mathbb{Q}\times\mathbb{R} \subset \mathbb{R}^2$

   **(d)** $S_4 = \mathbb{Z}^3 \subset \mathbb{R}^3$

4. If $\varnothing \neq S \subsetneq \mathbb{R}^N$ show that $\partial S \neq \varnothing$. Deduce that $\varnothing$ and $\mathbb{R}^N$ are the only simultaneously open and closed sets in $\mathbb{R}^N$.

   [Hint: if $x \in S$, $y \in S^c$, find a convex combination of $x$ and $y$ in $\partial U$.]

*Don't forget q. 5 ...*

5. Prove that if $U$ is an open subset of $\mathbb{R}$, then there exists a sequence of open intervals $\{I_j\}_{j\in J}$ (here, either $J$ is finite or $J = \mathbb{N}$) such that

$$I_j \cap I_k = \varnothing \text{ if } j \neq k \text{ in } J, \text{ and } U = \bigcup_{j\in J} I_j.$$

[Hint: if $x \in U$ let $a_x = \inf\{a : (a, x] \subset U\}$ and $b_x = \sup\{b : [x, b) \subset U\}$. Show that either $a_x = a_y$ and $b_x = b_y$ or $(a_x, b_x) \cap (a_y, b_y) = \varnothing$.]