MATH 247, FALL 2005

Assignment #1 Due: January 17.

1. Show that if $a < b$ in a complete ordered field $(\mathbb{O}, P)$, then there exists an *irrational* $\eta \in \mathbb{O} \setminus \mathbb{Q}$ with $a < \eta < b$.

2. Let P be be the order cone of $\mathbb{Q}$, and $\mathbb{P}$ the order cone of $\mathbb{R}$, where we view $\mathbb{R}$ as the family of Dedekind cuts.

   (a) Verify that for any A in $\mathbb{P}$, the element

   $$A^{-1} = \{q^{-1} : q \in \mathbb{Q} \setminus A \text{ and } q \neq \min(\mathbb{Q} \setminus A)\} \cup (-P) \cup \{0\}$$

   satisfies $A \cdot A^{-1} = A_1$ where $A_1 = \{q \in \mathbb{Q} : q < 1\}$.

   (b) Verify that $\mathbb{R}$ is complete. [Hint: $A \leq B \Leftrightarrow A \subseteq B$.]

   (c) Verify, for A, B and C, the distributive law: $A \cdot (B + C) = A \cdot B + A \cdot C$. [Hint: Consider case where $A, B, C \in \mathbb{P}$, first.]

3. **(a)** Let $(\mathbb{O}, P)$ be an ordered field. Show that if $0 \leq x < y$ in $\mathbb{O}$ and $n \in \mathbb{N}$, then $x^n < y^n$.

   (b) Show that if $a \geq 0$ in $\mathbb{R}$ and $n \in \mathbb{N}$, then there is a unique z in $\mathbb{R}$ such that

   $$z \geq 0 \text{ and } z^n = a.$$

   We write $z = \sqrt[n]{a}$.

   [Hint: You may want to use the binomial theorem and the factorization $y^n - x^n = (y - x) \sum_{k=1}^{n-1} y^{n-k} x^{k-1}$.]

4. Let $(\mathbb{O}, P)$ be an Archimedean ordered field. If $a < b$ in $\mathbb{O}$ we let $[a, b] = \{x \in \mathbb{O} : a \leq x \leq b\}$, and call this a *finite closed interval*.

   Show that if $\mathbb{O}$ satisisfies the *nested intervals property* (i.e. if $I_1 \supset I_2 \supset \ldots$ is a sequence of finite closed intervals in $\mathbb{O}$, then $\bigcap_{j=1}^{\infty} I_j \neq \varnothing$) then $\mathbb{O}$ must be *complete* (i.e. all bounded above sets have supremums).

   [Hint: If $\varnothing \neq S \subset \mathbb{O}$ is bounded above, let a_1 be any element which is *not* an upper bound for S, and b_1 be any upper bound for S. Bisect the interval $[a_1, b_1]$ to start an induction procedure which "narrows in" on a candidate for $\sup S$.]

   Don't forget q. 5 ...

5. Show, by way of examples, that a decreasing sequence $I_1, I_2, \ldots$ of intevals in $\mathbb{R}$ in may have $\bigcap_{j=1}^{\infty} I_j = \varnothing$ if

   (a) the intervals are of finite length but fail to be closed, or

   (b) the intervals are closed but not of finite length.